Math 54 Homework 9 Solutions

10.3-9 Write
$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}.$$

If $n \neq 0$, then
$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx}dx = \frac{1}{2\pi}\left[\frac{xe^{-inx}}{-in}\bigg|_{-\pi}^{\pi} - \frac{e^{-inx}}{(-in)^2}\bigg|_{-pi}^{\pi}\right] = \frac{(-1)^n}{-in}$$

If $n = 0$, then
$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} xdx = 0$$

Therefore the answer is
$$f(x) = \sum_{n\neq 0} \frac{(-1)^n}{-in}e^{inx}$$

To graph this, you can convert it to sines and cosines:
$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{-in}e^{inx} + \sum_{n=-1}^{-\infty} \frac{(-1)^n}{-in}e^{inx}$$
$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{-in}e^{inx} + \sum_{n=1}^{\infty} \frac{(-1)^{-n}}{-i(-n)}e^{-inx}$$
$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{-in}(e^{inx} - e^{-inx}) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n}\sin(nx)$$

10.3-10 Write
$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$$

If $n \neq 0$, then
$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -xe^{-inx}dx + \int_{0}^{\pi} xe^{-inx}dx\right]$$
$$= \frac{1}{2\pi}\left[\frac{-xe^{-inx}}{-in}\bigg|_{-\pi}^{0} + \frac{e^{-inx}}{(-in)^2}\bigg|_{-\pi}^{0} + \frac{xe^{-inx}}{-in}\bigg|_{0}^{\pi} - \frac{e^{-inx}}{(-in)^2}\bigg|_{0}^{\pi}\right]$$
$$= \frac{1}{2\pi}\frac{(2(1-(-1)^n))}{-n^2} = \begin{cases} -\frac{2}{\pi n^2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

And if $n = 0$ then
$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{\pi}{2}$$

so the answer is

$$f(x) = \frac{\pi}{2} + \sum_{n\neq 0} -\frac{1-(-1)^n}{\pi n^2} e^{inx}$$

or, simplified,

$$f(x) = \frac{\pi}{2} + \sum_{n \text{ odd}} -\frac{2}{\pi n^2} e^{inx}$$

If you're interested, here it is as sines and cosines:

$$f(x) = \frac{\pi}{2} + \sum_{\substack{n\geq 1 \\ n \text{ odd}}} -\frac{4}{\pi n^2} \cos(nx)$$

10.3-16 Write

$$f(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} a_n e^{inx}$$

If $n \neq 0$, then

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx = \frac{1}{2\pi} \left[-\int_{-\pi/2}^{0} e^{-inx} dx + \int_{0}^{\pi/2} e^{-inx} dx \right]$$

$$= -\frac{1}{2\pi} \, e^{inx} - in\Big|_{-\pi/2}^{0} + \frac{1}{2\pi} \left. \frac{e^{-inx}}{-in} \right|_{0}^{\pi/2}$$

$$= -\frac{1}{2\pi} \left(\frac{1-i^n}{-in} \right) + \frac{1}{2\pi} \left(\frac{(-i)^n - 1}{-in} \right)$$

$$= \frac{1}{2\pi i n} (2 - i^n(1 + (-1)^n))$$

If $n = 0$, then

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = 0$$

so the answer is

$$f(x) = \frac{2 - i^n(1+(-1)^n)}{2\pi i n} e^{inx}$$

This can be simplified a little, since

$$2 - i^n(1 + (-1)^n) = \begin{cases} 0 & n = 4k \text{ for some } k \\ 2 & n = 4k+1 \text{ for some } k \\ 4 & n = 4k+2 \text{ for some } k \\ 2 & n = 4k+3 \text{ for some } k \end{cases}$$

so

$$f(x) = \frac{1}{2\pi i n} \begin{bmatrix} 0 & n = 4k \text{ for some } k \\ 2 & n = 4k+1 \text{ for some } k \\ 4 & n = 4k+2 \text{ for some } k \\ 2 & n = 4k+3 \text{ for some } k \end{bmatrix} e^{inx}$$

Writing this as sines and cosines is:

$$f(x) = \sum_{n \geq 1} \frac{1}{\pi n} \begin{bmatrix} 0 & n = 4k \text{ for some } k \\ 2 & n = 4k+1 \text{ for some } k \\ 4 & n = 4k+2 \text{ for some } k \\ 2 & n = 4k+3 \text{ for some } k \end{bmatrix} \sin(nx)$$