Math 54 Homework 6 Solutions

5.3-4

$$A^k = \begin{pmatrix} 3 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2^k & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 4 \\ 1 & -3 \end{pmatrix} = \begin{pmatrix} -3 \cdot 2^k + 4 & 12 \cdot 2^k - 12 \\ -2^k + 1 & 4 \cdot 2^k - 3 \end{pmatrix}$$

5.3-6 The eigenvalues are 5 and 4. A basis for the 5-eigenspace is $(-2, 0, 1)$, $(0, 1, 0)$ and a basis for the 4-eigenspace is $(-1, 2, 0)$.

5.3-31 There are lots of correct answers, but here's one example: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. It's not diagonalizable because you can't find an eigenbasis for it because all of its eigenvectors lie on a single line. It's invertible because its determinant is 1.

5.3-32 There are lots of correct answers, but here's one example: the matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

has an eigenbasis consisting of $(1, 0)$ and $(1, 1)$. Therefore it is diagonalizable. It is not invertible because its determinant is 0.

5.5-1 The eigenvalues are the solutions to

$$(1 - \lambda)(3 - \lambda) + 2 = 0 \Rightarrow \lambda = \frac{4 \pm \sqrt{-4}}{2} = 2 \pm i$$

An eigenvector for the $2 + i$ eigenspace is an element in the kernel of

$$\begin{pmatrix} 1 - (2 + i) & -2 \\ 1 & 3 - (2 + i) \end{pmatrix} = \begin{pmatrix} -1 - i & -2 \\ 1 & 1 - i \end{pmatrix}$$

so $(1 - i, -1)$ is such an eigenvector. An eigenvector for the $2 - i$ eigenspace is an element of the kernel of

$$\begin{pmatrix} 1 - (2 - i) & -2 \\ 1 & 3 - (2 - i) \end{pmatrix} = \begin{pmatrix} -1 + i & -2 \\ 1 & 1 + i \end{pmatrix}$$

so $(1 + i, -1)$ is such an eigenvector. An eigenbasis of $\mathbb{C}^2$ is the basis consisting of $(1 - i, -1)$ and $(1 + i, -1)$.

5.5-2 The eigenvalues satisfy $\lambda^2 - 6\lambda + 10 = 0$ and so are $(3 \pm i)/2$. An eigenvector for the $3 + i$ eigenspace is an element of the kernel of

$$\begin{pmatrix} 5 - (3 + i) & -5 \\ 1 & 1 - (3 + i) \end{pmatrix} = \begin{pmatrix} 2 - i & -5 \\ 1 & -2 - i \end{pmatrix}$$

so $(2+i, 1)$ is such an eigenvector. An eigenvector for the $3-i$ eigenspace is an element ofthe kernel of

$$\begin{pmatrix} 5-(3-i) & -5 \\ 1 & 1-(3-i) \end{pmatrix} = \begin{pmatrix} 2+i & -5 \\ 1 & -2+i \end{pmatrix}$$

so $(2-i, 1)$ is such an eigenvector. An eigenbasis of $\mathbb{C}^2$ is the basis consisting of $(2+i, 1)$ and $(2-i, 1)$.

3. Let

$$A = \begin{pmatrix} 0 & 1 \\ -q/p & 1/p \end{pmatrix}$$

Since

$$A \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix} = \begin{pmatrix} a_{n+1} \\ a_{n+2} \end{pmatrix}$$

then

$$A^N \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix} = \begin{pmatrix} a_{n+N} \\ a_{a+N+1} \end{pmatrix}.$$

Therefore a_{n+N} can be computed as the first coordinate of $A^N \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix}$. In particular, a_0 is the first coordinate of $A^{-n} \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix}$ and a_{100} is the first coordinate of $A^{100-n} \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix}$. These two equations give two equations in the two unknown variables a_n and a_{n+1}. From these two equations, you can solve for a_n. Therefore, one must first compute A^N.

The eigenvalues of A satisfy $(-\lambda)(1/p - \lambda) + q/p = 0$ so

$$\lambda = \frac{\frac{1}{p} \pm \sqrt{\frac{1}{p^2} - \frac{4q}{p}}}{2} = \frac{\frac{1}{p} \pm \sqrt{\frac{1}{p^2} - \frac{4(1-p)}{p}}}{2} = 1, \ \frac{1-p}{p} = 1, \frac{q}{p}$$

An eigenvector for the 1-eigenvalue is a vector in the kernel of

$$\begin{pmatrix} -1 & 1 \\ -\frac{q}{p} & \frac{1}{p} - 1 \end{pmatrix}$$

so $(1, 1)$ is such an eigenvector. An eigenvector for the q/p eigenvalue is a vector in the kernel of

$$\begin{pmatrix} -\frac{q}{p} & 1 \\ -\frac{q}{p} & \frac{1}{p} - \frac{q}{p} \end{pmatrix}$$

so $(1, q/p)$ is such an eigenvector. Therefore

$$A^N = PD^N P^{-1}$$

where

$$P = \begin{pmatrix} 1 & 1 \\ 1 & q/p \end{pmatrix}, \; D^N = \begin{pmatrix} 1 & 0 \\ 0 & (q/p)^N \end{pmatrix}, \; P^{-1} = \frac{1}{\frac{q}{p}-1}\begin{pmatrix} q/p & -1 \\ -1 & 1 \end{pmatrix}$$

Therefore

$$A^N = \frac{1}{\frac{q}{p}-1}\begin{pmatrix} \frac{q}{p} - \left(\frac{q}{p}\right)^N & -1 + \left(\frac{q}{p}\right)^N \\ \frac{q}{p} - \left(\frac{q}{p}\right)^{N+1} & -1 + \left(\frac{q}{p}\right)^{N+1} \end{pmatrix}$$

It will be helpful to write $\lambda := \frac{q}{p}$ so

$$A^N = \frac{1}{\lambda - 1}\begin{pmatrix} \lambda - \lambda^N & -1 + \lambda^N \\ \lambda - \lambda^{N+1} & -1 + \lambda^{N+1} \end{pmatrix}$$

Then

$$A^N \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix} = \frac{1}{\lambda - 1}\begin{pmatrix} (\lambda - \lambda^N)a_n + (-1 + \lambda^N)a_{n+1} \\ (\lambda - \lambda^{N+1})a_n + (-1 + \lambda^{N+1})a_{n+1} \end{pmatrix}$$

Therefore

$$a_{n+N} = \frac{1}{\lambda - 1}\left((\lambda - \lambda^N)a_n + (-1 + \lambda^N)a_{n+1}\right)$$

so by setting $N = -n$ one gets

$$0 = a_0 = \frac{1}{\lambda - 1}\left((\lambda - \lambda^{-n})a_n + (-1 + \lambda^{-n})a_{n+1}\right)$$

and by setting $N = 100 - n$ one gets

$$1 = a_{100} = \frac{1}{\lambda - 1}\left((\lambda - \lambda^{100-n})a_n + (-1 + \lambda^{100-n})a_{n+1}\right)$$

The first equation implies that

$$a_{n+1} = \frac{\lambda - \lambda^{-n}}{1 - \lambda^{-n}} a_n$$

and then the second equation implies that

$$(\lambda - 1)(1 - \lambda^{-n}) = [(\lambda - \lambda^{100-n})(1 - \lambda^{-n}) + (-1 + \lambda^{100-n})(\lambda - \lambda^{-n})]a_n$$

so that

$$(\lambda - 1)(1 - \lambda^{-n}) = (-\lambda^{100-n} - \lambda^{-n+1} + \lambda^{100-n+1} + \lambda^{-n})a_n$$

so

$$(\lambda - 1)(1 - \lambda^{-n}) = (\lambda - 1)(\lambda^{100-n} - \lambda^{-n})a_n$$

$$\Rightarrow a_n = \frac{1 - \lambda^n}{1 - \lambda^{100}}$$

Plugging in $n = 80$ and $\lambda = .51/.49$ gives approximately .439, so there's about a 44% chance you win the game starting at $n = 80$.