Math 54 Homework 3 Solutions

4.1-13 (a) $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a set containing three vectors. $\mathbf{w}$ is not one of these vectors. Therefore $\mathbf{w}$ is not in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(b) There are infinitely many vectors in the span.

(c) Yes, $\mathbf{w} = \mathbf{v}_1 + \mathbf{v}_2$, so it's a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

4.1-14 Want to solve

$$a_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + a_2 \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + a_3 \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 7 \end{pmatrix}$$

which involves row reducing

$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & 8 \\ 0 & 1 & 2 & 4 \\ -1 & 3 & 6 & 7 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 4 & 8 \\ 0 & 1 & 2 & 4 \\ 0 & 5 & 10 & 15 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 4 & 8 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

at which point the last row implies that $0 = -1$, a contradiction. Therefore there is no solution (a_1, a_2, a_3). Therefore $\mathbf{w}$ is not in subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

4.1-35 As before want to solve

$$a_1 \begin{pmatrix} 8 \\ -4 \\ -3 \\ 9 \end{pmatrix} + a_2 \begin{pmatrix} -4 \\ 3 \\ -2 \\ -8 \end{pmatrix} + a_3 \begin{pmatrix} -7 \\ 6 \\ 5 \\ -18 \end{pmatrix} = \begin{pmatrix} 9 \\ -4 \\ -4 \\ 7 \end{pmatrix}$$

which involves row reducing

$$\left(\begin{array}{ccc|c} 8 & -4 & -7 & 9 \\ -4 & 3 & 6 & -4 \\ -3 & -2 & -5 & -4 \\ 9 & -8 & -18 & 7 \end{array}\right)$$

this is a bit annoying to start, but notice that the first row minus the second row plus the third row is almost the fourth row. So multiply the second row times -1 then subtract the first three rows from the last. This gives

$$\left(\begin{array}{ccc|c} 8 & -4 & -7 & 9 \\ 4 & -3 & -6 & 4 \\ -3 & -2 & -5 & -4 \\ 0 & 1 & 0 & -2 \end{array}\right)$$

You can then add multiples of the last row to the other rows to clear out the second column:

$$\left(\begin{array}{ccc|c} 8 & 0 & -7 & 1 \\ 4 & 0 & -6 & -2 \\ -3 & 0 & -5 & -8 \\ 0 & 1 & 0 & -2 \end{array}\right)$$

Add -4 times the second row to the first to get

$$\left(\begin{array}{ccc|c} 0 & 0 & 5 & 5 \\ 2 & 0 & -3 & -1 \\ -3 & 0 & -5 & -8 \\ 0 & 1 & 0 & -2 \end{array}\right)$$

this says that $a_2 = -2$ and $5a_3 = 5$ so $a_3 = 1$. Therefore you can determine that $a_1 = 1$:

$$\mathbf{w} = \mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3$$

4.3-1 Basis

4.3-2 Not a basis, not linearly independent, does not span

4.3-3 Not a basis, not linearly independent, does not span

4.3-4 A basis, linearly independent, spanning

4.3-5 Not a basis, not linearly independent, spanning

4.3-6 Not a basis, linearly independent, not spanning

4.3-11 This plane is parametrized by

$$\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} -2x_2 + x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

The two vectors $(-2, 1, 0)$ and $(-1, 0, 1)$ therefore span the plane. Since they are not multiples of each other they are linearly independent and hence form a basis for the plane.

4.3-12 A basis of a line is one vector. In this example, $(1, 5)$ is a basis.

4.3-15 Remember you can always add vectors to a linearly independent set to get a basis. Start with the first two vectors, which are clearly linearly independent:

$$\begin{pmatrix} 1 \\ 0 \\ -3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ -3 \end{pmatrix}$$

adding in the next vector might ruin the linear independence. One can check this by row reducing

$$\begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & -4 \\ -3 & 2 & 1 \\ 2 & -3 & 6 \end{pmatrix}$$

which shows that the third vector is in the span of the first two. One can then try to add in the fourth vector. Row reducing

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ -3 & 2 & -8 \\ 2 & -3 & 7 \end{pmatrix}$$

shows that the fourth vector is linearly independent from the first two. One can try to add in the fifth vector to these three. This might ruin the linear independence. One can check this by row reducing

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -3 & 1 \\ -3 & 2 & -8 & -6 \\ 2 & -3 & 7 & 9 \end{pmatrix}$$

which shows that the fifth vector is the in the span of the first, second, and fourth. Therefore a basis consists of the first, second, and fourth vectors.

An easier way to do this, discussed in class after the homework was due, is to row reduce

$$\begin{pmatrix} 1 & 0 & -3 & 1 & 2 \\ 0 & 1 & -4 & -3 & 1 \\ -3 & 2 & 1 & -8 & -6 \\ 2 & -3 & 6 & 7 & 9 \end{pmatrix}$$

and take the columns corresponding to the pivot variables.

4.3-16 Similar to the last question: $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$.

4.3-19 One can remove vectors from a spanning set until one gets a linearly independent set. The resulting set of vectors is a basis. Since the three vectors are linearly dependent, they do not form a basis. Remove $\mathbf{v}_3$ to get two vectors which are not multiples of one another and are therefore linearly independent. Therefore $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis of the span of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.