Math 54 Homework 10 Solutions

1. (a) $f(x) = e^{-4ix} + e^{-2ix} + e^{2ix} + e^{4ix}$ is already a Fourier series for f. Since $u(x+2\pi, t) = u(x,t)$ for each times t, write

$$u(x,t) = \sum_{n=-\infty}^{\infty} a_n(t) e^{inx}.$$

Plugging this expression for u into

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

gives

$$a_n'(t) = -n^2 a_n(t)$$

which implies that

$$a_n(t) = e^{-n^2 t} a_n(0).$$

Here $a_n(0)$ is the nth Fourier coefficient of f, so it's 1 is $n \in \{\pm 2, \pm 4\}$ and 0 otherwise. Therefore

$$u(x,t) = e^{4ix}e^{-16t} + e^{2ix}e^{-4t} + e^{-2ix}e^{-4t} + e^{-4ix}e^{-16t}$$

$$u(0,t) = 2e^{-16t} + 2e^{-4t}$$

(b) From homework 9:

$$f(x) = \frac{\pi}{2} + \sum_{n \text{ odd}} \left(-\frac{2}{\pi n^2}\right) e^{inx}$$

therefore

$$a_n(0) = \begin{cases} \frac{\pi}{2} & \text{if } n = 0 \\ -\frac{2}{\pi n^2} & \text{if } n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

so

$$u(x,t) = \frac{\pi}{2} + \sum_{n \text{ odd}} \left(-\frac{2}{\pi n^2}\right) e^{-n^2 t} e^{inx}$$

2. Write

$$f(x) = \sum_n b_n e^{inx}$$

$$g(x) = \sum_n c_n e^{inx}$$

(a)

$$f(x) = \cos(x) = \frac{e^{ix} + e^{-ix}}{2} \Rightarrow b_n = \begin{cases} 1/2 & \text{if } n = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

$$g(x) = \sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \Rightarrow c_n = \begin{cases} 1/2i & \text{if } n = 1 \\ -1/2i & \text{if } n = -1 \\ 0 & \text{otherwise} \end{cases}$$

So

$$u(x,t) = b_0 + c_0 t + \sum_{n \neq 0} \left(b_n \cos(nt) + \frac{c_n}{n} \sin(nt) \right) e^{inx}$$

$$\Rightarrow u(x,t) = \left(\frac{\cos(t)}{2} + \frac{\sin(t)}{2i} \right) e^{ix} + \left(\frac{\cos(t)}{2} - \frac{\sin(t)}{2i} \right) e^{-ix}$$

(this, in fact, simplifies to $\cos(x-t)$).

(b)

$$f(x) = 1 \Rightarrow b_n = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

And from homework 9:

$$c_n = \begin{cases} \frac{(-1)^n}{-in} & \text{if } n \neq 0 \\ 0 & \text{if } n = 0 \end{cases}$$

Therefore

$$u(x,t) = 1 + \sum_{n \neq 0} \left(\frac{(-1)^n}{-in^2} \sin(nt) \right) e^{inx}$$

3. (a)

$$a_n''(t) = -n^2 a_n(t) + a_n \Rightarrow a_n(t) = a_n(0) e^{(1-n^2)t} = b_n e^{(1-n^2)t}$$

$$\Rightarrow u(x,t) = \sum_{n=-\infty}^{\infty} b_n e^{(1-n^2)t} e^{inx}$$

(b) All the terms in $u(x,t)$ go to zero except those corresponding to $n = \pm 1$ and 0. The $n = \pm 1$ terms stay the same as $t \to \infty$ and the the $n = 0$ term goes to ∞. Therefore $u(x,t)$ becomes infinite as $t \to \infty$.

4. As always in this class, we assume that u is periodic in x with period 2π: $u(x + 2\pi, t) = u(x,t)$. Therefore

$$u(x,t) = \sum_n a_n(t) e^{inx}$$

for some coefficients $a_n(t)$. Plug this expression for u into the differential equation to get

$$a_n''(t) = -n^2 a_n(t) + 5a_n(t)$$
$$\Rightarrow a_n(t) = Ae^{\sqrt{5-n^2}t} + Be^{-\sqrt{5-n^2}t}$$
$$a_n'(t) = \sqrt{5-n^2}(Ae^{\sqrt{5-n^2}t} - Be^{-\sqrt{5-n^2}t})$$
$$\Rightarrow A = \frac{1}{2}\left(a_n(0) + \frac{a_n'(0)}{\sqrt{5-n^2}}\right)$$
$$\Rightarrow B = \frac{1}{2}\left(a_n(0) - \frac{a_n'(0)}{\sqrt{5-n^2}}\right)$$

$$\Rightarrow u(x,t) = \sum_n \left(\frac{1}{2}\left(a_n(0) + \frac{a_n'(0)}{\sqrt{5-n^2}}\right)e^{\sqrt{5-n^2}t} + \frac{1}{2}\left(a_n(0) - \frac{a_n'(0)}{\sqrt{5-n^2}}\right)e^{-\sqrt{5-n^2}t}\right)e^{inx}$$

$$= \sum_n \left(a_n(0)\left(\frac{e^{\sqrt{5-n^2}t} + e^{-\sqrt{5-n^2}t}}{2}\right) + \frac{a_n'(0)}{\sqrt{5-n^2}}\left(\frac{e^{\sqrt{5-n^2}t} - e^{-\sqrt{5-n^2}}}{2}\right)\right)e^{inx}$$

In the case where

$$f(x) = \cos(3x) = \frac{1}{2}e^{ix} + \frac{1}{2}e^{-ix}$$

and

$$g(x) = \sin(x) + \sin(2x) = \frac{1}{2i}e^{ix} - \frac{1}{2i}e^{-ix} + \frac{1}{2i}e^{2ix} - \frac{1}{2i}e^{-2ix}$$

Then $b_{\pm} = \frac{1}{2}$, $c_1 = \frac{1}{2i}$, $c_{-1} = -\frac{1}{2i}$, $c_2 = \frac{1}{2i}$, $c_{-2} = -\frac{1}{2i}$, and all other Fourier coefficients of f and g are zero. Therefore the sum of $u(x,t)$ is a finite sum:

$$u(x,t) = \left(\frac{e^{2it} + e^{-2it}}{4}\right)\left(e^{3ix} + e^{-3ix}\right) + \left(\frac{e^{2t} - e^{-2t}}{8i}\right)\left(e^{ix} - e^{-ix}\right) + \left(\frac{e^{t} - e^{-t}}{4i}\right)\left(e^{2ix} - e^{-2ix}\right)$$