Given a function on the real numbers, one can often write it as a series of powers, for example

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$$

In general, the formula is:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

These series are called Taylor series. In this section, we'll learn about a different kind of series for periodic functions, called Fourier series.

Recall f is periodic with period 2π if $f(x + 2\pi) = f(x)$. For example, $e^{ix} = \cos(x) + i\sin(x)$ is periodic with period 2π. Also note that $e^{inx} = (e^{ix})^n$ is a power of this function. These functions are called "complex exponentials".

If f is periodic with period 2π, it can be written as an infinite sum of complex exponentials:[1]

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$$

where the coefficients a_n can be computed exactly. To compute the coefficients, note that[2]

$$\int_0^{2\pi} f(x)e^{-imx}dx = \int_0^{2\pi} \sum_{n=-\infty}^{\infty} a_n e^{i(n-m)x}dx = \sum_{n=-\infty}^{\infty} a_n \int_0^{2\pi} e^{i(n-m)x}dx$$

and

$$\int_0^{2\pi} e^{i(n-m)x}dx = \begin{cases} 2\pi & n = m \\ 0 & \text{otherwise} \end{cases}$$

so in the infinite sum, only the term corresponding to m is nonzero. Therefore

$$2\pi a_m = \int_0^{2\pi} f(x)e^{-imx}dx$$

[1] There are some footnotes that should be added here. You can devise functions for which this sum does not exist or does not converge, but we will not deal with these. For most practical purposes, the sum here exists. More seriously, if you change f at a finite number of points, the series on the right does not change. Therefore the series on the right converges to f except at perhaps a finite number of points. These details are important in the full study of these series, but will not be very important to us. Just know the use of an "equals" sign here should be used cautiously. In general, the series converges to the correct values of f at points where f is continuous.

[2] Assume here that you can interchange the sum and the integral. This doesn't work for all functions, but it works for the functions we'll care about.

so

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx.$$

Note that this formula also works with integrand bounds $-\pi$ and π since f is period of period 2π:

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

The series $\sum_{n=-\infty}^{\infty} a_n e^{inx}$ is called a Fourier series and the coefficients a_n are called Fourier coefficients.

For example, if

$$f(x) = \begin{cases} 1 & 0 \le x < \pi \\ 0 & \pi \le x < 2\pi \end{cases}$$

and then you translate the graph of f to make it periodic, then

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx = \frac{1}{2\pi} \int_0^{\pi} e^{-inx} dx$$

$$= \frac{1}{2\pi} \left. \frac{e^{-inx}}{-in} \right|_0^{\pi} = \frac{1}{2\pi} \frac{e^{i\pi n} - 1}{-in} = \frac{(e^{i\pi})^n - 1}{-2\pi i n} = \frac{(-1)^n - 1}{-2\pi i n}$$

If n is even, note that this vanishes. This computation only works for $n \neq 0$, since in at one point we divided by n. One must deal with the $n = 0$ case separately:

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-i \cdot 0 \cdot x} dx = \frac{1}{2\pi} \int_0^{2\pi} f(x) dx = \frac{1}{2}$$

Therefore

$$f(x) = \frac{1}{2} + \sum_{n \text{ odd}} \frac{-i}{\pi n} e^{inx}$$

You can turn this series into a series involving sines and cosines by splitting it up into positive and negative parts:

$$= \frac{1}{2} + \sum_{\substack{n \ge 1 \\ n \text{ odd}}} \frac{-i}{\pi n} e^{inx} + \sum_{\substack{n \le -1 \\ n \text{ odd}}} \frac{-i}{\pi n} e^{inx}$$

changing variables $n \to -n$ in the second sum

$$= \frac{1}{2} + \sum_{\substack{n \ge 1 \\ n \text{ odd}}} \frac{-i}{\pi n} e^{inx} + \sum_{\substack{n \ge 1 \\ n \text{ odd}}} \frac{-i}{\pi(-n)} e^{i(-n)x}$$

combining the last two summations

$$= \frac{1}{2} \sum_{\substack{n \ge 1 \\ n \text{ odd}}} \frac{-i}{\pi n} (e^{inx} - e^{-inx})$$

and using $e^{inx} = \cos(nx) + i\sin(nx)$ and $e^{-inx} = \cos(nx) - i\sin(nx)$:

$$= \frac{1}{2} + \sum_{\substack{n\geq 1 \\ n \text{ odd}}} \frac{2}{\pi n}\sin(nx).$$

In this form you can use a computer to graph the partial sums of the series and visually see that they converge to the appropriate function.

Up to now in class we've discussed differential equations involving functions of a single variable:

$$\frac{df}{dx} = f \Rightarrow f(x) = Ce^x$$

and possibly with some initial conditions:

$$f(0) = 2 \Rightarrow f(x) = 2e^x.$$

These sorts of differential equations are called "ordinary differential equations". If the function involves more than one variable then the differential equation is called a "partial differential equation". For example if $u(x,t)$ is a function of two variables, then[3]

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

is a partial differential equation. For the PDEs we'll be dealing with, it will be helpful to think of t as a time parameter, and $u(x,t)$ a family of single variable functions varying in time. With this notion, an initial condition is of the form $u(x,0) = f(x)$. Therefore one might want to solve the differential equation with initial condition:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$
$$u(x,0) = f(x)$$

for some given fixed single variable function $f(x)$. This particular partial differential equation is called the heat equation. If u represents the temperature at position x and time t, then its behavior follows this equation. f represents the initial temperature distribution at time $t = 0$[4]

In this course, we'll just focus on the case where u (and hence f) is periodic (with period 2π) in its x coordinate: $u(x + 2\pi, t) = u(x,t)$. Then one way to

[3]Here $\frac{\partial u}{\partial t}$ means "differentiate u with respect to t, treating all other variables as constants." For example if $u(x,t) = tx^2 + \cos(x)$, then $\frac{\partial u}{\partial t} = x^2$.

[4]Strictly speaking, since we live in three-dimensional space, the real heat equation is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

and the "heat equation" given here is its restriction to a single direction.

solve the heat equation is to write down a Fourier series for u at each time t. Since u is different at each time t, the Fourier coefficients depend on t:

$$u(x,t) = \sum_{-\infty}^{\infty} a_n(t)e^{inx}$$

Then

$$\frac{\partial u}{\partial t} = \sum_{-\infty}^{\infty} a_n'(t)e^{inx}$$

$$\frac{\partial^2 u}{\partial x^2} = \sum_{-\infty}^{\infty} a_n(t)(in)^2 e^{inx}$$

so, plugging these into the heat equation, one gets

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \Rightarrow \sum_{-\infty}^{\infty} a_n'(t)e^{inx} = \sum_{-\infty}^{\infty} a_n(t)(in)^2 e^{inx}$$

and equating the coefficients of the complex exponentials on each side implies that, for each n,

$$a_n'(t) = -n^2 a_n(t)$$

Therefore $a_n(t) = Ce^{-n^2 t}$ for some constant C. Plugging in $t = 0$ shows that $C = a_n(0)$, so $a_n(t) = a_n(0)e^{-n^2 t}$ so that

$$u(x,t) = \sum_{-\infty}^{\infty} a_n(0)e^{-n^2 t}e^{inx}.$$

Therefore

$$f(x) = u(x,0) = \sum_{-\infty}^{\infty} a_n(0)e^{inx}$$

so $a_n(0)$ is the nth Fourier coefficient of f.

For example, if $f(x)$ is the same function as earlier:

$$f(x) = \begin{cases} 1 & 0 \le x < \pi \\ 0 & \pi \le x < 2\pi \end{cases}$$

(translated to be 2π-periodic) then

$$a_n(0) = \begin{cases} \frac{1}{2} & n = 0 \\ \frac{-i}{\pi n} & n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

so that

$$u(x,t) = \frac{1}{2} + \sum_{n \text{ odd}} \frac{-i}{\pi n} e^{-n^2 t} e^{inx}$$

As before you can write this in terms of sines and cosines if you so choose:

$$u(x,t) = \frac{1}{2} + \sum_{\substack{n\geq 1 \\ n \text{ odd}}} \frac{2}{\pi n} e^{-n^2 t} \sin(nx)$$

Another important PDE is the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = f(x)$$

$$\frac{\partial u}{\partial t}(x,0) = g(x)$$

Since there are two time derivatives in the differential equation, it turns out that you have to specify initial conditions both for u and for its time derivative. Again, we'll only solve this equation in the case where u is periodic in the x-direction: $u(x+2\pi,t) = u(x,t)$. In this case, f and g are periodic and so they admit Fourier series:

$$f(x) = \sum_{n=\infty}^{\infty} b_n e^{inx}$$

$$g(x) = \sum_{n=\infty}^{\infty} c_n e^{inx}$$

for some constants b_n and c_n. To solve the wave equation, as before write out a Fourier series for u at each time t:

$$u(x,t) = \sum_{-\infty}^{\infty} a_n(t) e^{inx}$$

Then

$$\frac{\partial^2 u}{\partial t^2} = \sum_{-\infty}^{\infty} a_n''(t) e^{inx}$$

$$\frac{\partial^2 u}{\partial x^2} = \sum_{-\infty}^{\infty} a_n(t)(in)^2 e^{inx}$$

and, plugging these into the wave equation shows that

$$a_n''(t) = -n^2 a_n(t) \Rightarrow a_n(t) = Ae^{int} + Be^{-int}$$

for some constants A and B. Furthermore,

$$a_n'(t) = inAe^{int} - inBe^{-int}$$

so that

$$a_n(0) = A + B$$

$$\frac{a_n'(0)}{in} = A - B$$

so that

$$A = \frac{1}{2}\left(a_n(0) + \frac{a_n'(0)}{in}\right)$$
$$B = \frac{1}{2}\left(a_n(0) - \frac{a_n'(0)}{in}\right)$$

But since $u(x,0) = f(x)$, then $a_n(0) = b_n$. And since $\frac{\partial u}{\partial t}(x,0) = g(x)$, then $a_n'(0) = c_n$. Therefore

$$A = \frac{1}{2}\left(b_n + \frac{c_n}{in}\right)$$
$$B = \frac{1}{2}\left(b_n - \frac{c_n}{in}\right)$$

so that

$$a_n(t) = \frac{1}{2}\left(b_n + \frac{c_n}{in}\right)e^{int} + \frac{1}{2}\left(b_n - \frac{c_n}{in}\right)e^{-int}$$
$$= b_n\cos(nt) + \frac{c_n}{n}\sin(nt)$$

Note that when $n = 0$, this analysis doesn't apply since I divided by 0. Therefore you should treat that case separately:

$$a_0''(t) = 0 \Rightarrow a_0(t) = A + Bt$$

where here $A = a_0(0)$ and $B = a_0'(0)$. Therefore

$$u(x,t) = b_0 + c_0 t + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left(b_n\cos(nt) + \frac{c_n}{n}\sin(nt)\right)e^{inx}$$

Given f and g, you can compute the coefficients b_n and c_n, and hence compute the solution $u(x,t)$. For example, if f is as before:

$$f(x) = \begin{cases} 1 & 0 \leq x < \pi \\ 0 & \pi \leq x < 2\pi \end{cases}$$

(extended to the whole real line in a way that makes it 2π periodic) and $g(x) = 0$, then

$$b_n = \begin{cases} \frac{1}{2} & n = 0 \\ \frac{-i}{\pi n} & n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$
$$c_n = 0$$

so that

$$u(x,t) = \frac{1}{2} + \sum_{n \text{ odd}} \frac{-i}{\pi n}\cos(nt)e^{inx}.$$

As usual, this can be written in terms of sines and cosines with the usual trick of splitting the sum into positive n and negative n, then changing variables $n \to -n$ on the negative part:

$$u(x,t) = \frac{1}{2} + \sum_{n \text{ odd}} \frac{2}{\pi n} \cos(nt)\sin(nx)$$

In the solution of the heat equation, the nth Fourier coefficient slowly decreases in size as time increases. In the solution to the wave equation, it oscillates back and forth in magnitude.

Here's a simpler partial differential equation, the transport equation:

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}$$

$$u(x,0) = f(x)$$

Again, suppose that $u(x+2\pi,t) = u(x,t)$. In particular f is periodic so

$$f(x) = \sum_{n=-\infty}^{\infty} b_n e^{inx}$$

for some coefficients b_n. You can write $u(x,t)$ in terms of the coefficients b_n by expanding $u(x,t)$ as a Fourier series at each time t:

$$u(x,t) = \sum_{n=-\infty}^{\infty} a_n(t) e^{inx}$$

plugging this expression for u into both sides of the transport equation:

$$\frac{\partial u}{\partial t} = \sum_{n=-\infty}^{\infty} a_n'(t) e^{inx}$$

$$\frac{\partial u}{\partial x} = \sum_{n=-\infty}^{\infty} ina_n(t) e^{inx}$$

setting the two sides equal to one another

$$\sum_{n=-\infty}^{\infty} a_n'(t) e^{inx} = \sum_{n=-\infty}^{\infty} ina_n(t) e^{inx}$$

equating coefficients

$$a_n'(t) = ina_n(t)$$

and solving

$$a_n(t) = a_n(0)e^{int} = b_n e^{int}$$

Therefore

$$u(x,t) = \sum_{n=-\infty}^{\infty} b_n e^{int} e^{inx}$$

This is the solution. In fact, you can simplify things a bit:

$$u(x,t) = \sum_{n=-\infty}^{\infty} b_n e^{in(x+t)} = f(x+t)$$

so the solution to the transport equation is just a family of shifted copies of f.