Math 54 Final Practice 2 Solutions

1. There are many ways to solve this, one of which is row reduction

$$\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 1 & -1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right)$$

so only solution is $x = 1$, $y = 1$.

2. The columns span the image and in this case the columns are linearly independent, so they form a basis. Therefore a basis for the image is

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

The kernel is 1-dimensional, so a basis for the kernel consists of a single nonzero vector. Since the third column is 0, then $(0, 0, 1)$ is in the kernel, and so gives a basis for the kernel.

3.
$$\det \begin{pmatrix} -13 - \lambda & -10 \\ 15 & 12 - \lambda \end{pmatrix} = (\lambda + 3)(\lambda - 2)$$

so the eigenvalues are -3 and 2. If $\lambda = -3$, an eigenvector is a vector in the kernel of

$$\begin{pmatrix} -10 & -10 \\ 15 & 15 \end{pmatrix}$$

one of which is $(1, -1)$. If $\lambda = 2$, an eigenvector is a vector in the kernel of

$$\begin{pmatrix} -15 & -10 \\ 15 & 10 \end{pmatrix}$$

an example of which is $(-2, 3)$. Therefore if

$$P = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$$

and

$$D = \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix}$$

then

$$A = PDP^{-1}$$

and hence

$$A^n = PD^nP^{-1}$$

Note that

$$P^{-1} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$$

so that

$$PD^nP^{-1} = \begin{pmatrix} 3 \cdot (-3)^n - 2 \cdot 2^n & 2 \cdot (-3)^n - 2 \cdot 2^n \\ -3 \cdot (-3)^n + 3 \cdot 2^n & -2 \cdot (-3)^n + 3 \cdot 2^n \end{pmatrix}$$

4. $A = I$ and $B = -I$ works

5. Consider any matrix like

$$\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$$

where the $*$s are replaced by numbers. There are infinitely many of these and their only eigenvalue is 1.

6. The lengths of the sides of the triangle are the lengths of the vectors that point for one vertex to another.

$$\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

and the length of this vector is $\sqrt{6}$.

$$\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

and the length of this vector is $\sqrt{6}$.

$$\begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}$$

and the length of this vector is $3\sqrt{2}$. Therefore the side lengths of the triangle are $\sqrt{6}$, $\sqrt{6}$, and $3\sqrt{2}$.

7. (updated 8/14) Vectors in P are the same vectors in the kernel of

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

By row reduction, these are vectors of the form

$$\begin{pmatrix} w \\ -z - w \\ z \\ w \end{pmatrix}$$

for some values of w and z. Therefore

$$v_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

forms a basis of P. Apply Gram-Schmidt to (v_1, v_2) as follows:

$$u_1 = \frac{1}{\|v_1\|}v_1 = \begin{pmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$w_2 = v_2 - (v_2 \cdot u_1)u_1 = \begin{pmatrix} 1 \\ -1/2 \\ -1/2 \\ 1 \end{pmatrix}$$

$$u_2 = \frac{1}{\|w_2\|}w_2 = \frac{2}{\sqrt{10}} \begin{pmatrix} 1 \\ -1/2 \\ -1/2 \\ 1 \end{pmatrix}$$

(u_1, u_2) is an orthonormal basis of P.

$\dim(P) = 2$ so $\dim(P^\perp) = \dim(\mathbb{R}^4) - \dim(P) = 2$. Therefore a basis of $P^\perp$ consists of two vectors in $P^\perp$ that do not lie on the same line. Vectors in $P^\perp$ are vectors (a, b, c, d) such that $ax + by + cz + dw = 0$ for all (x, y, z, w) in P. Vectors in P are vectors (x, y, z, w) such that $x + y + z = 0$ and $y + z + w = 0$. Therefore you want to find two vectors in (a, b, c, d) such that $ax + by + cz + dw = 0$ if $x + y + z = 0$ and $y + z + w = 0$. Two clear choices are $v_3 = (1, 1, 1, 0)$ and $v_4 = (0, 1, 1, 1)$.

Apply Gram-Schmidt to (v_3, v_4) as follows:

$$u_3 = \frac{v_3}{\|v_3\|} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$w_4 = v_4 - (v_4 \cdot u_3)u_3 = \begin{pmatrix} -2/3 \\ 1/3 \\ 1/3 \\ 1 \end{pmatrix}$$

$$u_4 = \frac{w_4}{\|w_4\|} = \frac{3}{\sqrt{15}} \begin{pmatrix} -2/3 \\ 1/3 \\ 1/3 \\ 1 \end{pmatrix}$$

Then (u_3, u_4) forms an orthonormal basis of $P^\perp$.

8. $\{v_1, v_2, v_3\}$ is linearly independent if and only if the matrix whose columns are v_1, v_2, and v_3 is injective. If this matrix is square, it's injective if and

only if it's invertible. Therefore

$$\begin{pmatrix} a \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ a \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ a \end{pmatrix}$$

is a linearly independent set of vectors if and only if the matrix

$$\begin{pmatrix} a & 0 & 1 \\ 1 & a & 0 \\ 0 & 1 & a \end{pmatrix}$$

is invertible, which occurs if and only if its determinant is nonzero. Its determinant is a^3+1, so if $a = -1$ then the vectors are linearly dependent and otherwise they're linearly independent. Therefore if $a \neq -1$ they're linearly independent and if $a \neq -1$ they also span.

9. Recall that the closest point on V to $(1, 0, 0, 0)$ is the orthogonal projection of $(1, 0, 0, 0)$ onto V. An orthonormal basis of V is

$$u_1 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix}, \ u_2 = \begin{pmatrix} 0 \\ 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

and the orthogonal projection of $e_1 = (1, 0, 0, 0)$ onto V is

$$(e_1 \cdot u_1)u_1 + (e_2 \cdot u_2)u_2 = \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \\ 0 \end{pmatrix}$$

10. Row reduce

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right)$$

so the inverse is

$$\begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

11. (a) The unit vector in the direction of y is

$$u = (1/\sqrt{n}, \ldots, 1/\sqrt{n}).$$

Note that

$$x \cdot u = \begin{cases} 0 & n \text{ even} \\ 1/\sqrt{n} & n \text{ odd} \end{cases}$$

The projection of x onto the span of y is $(x \cdot u)u$, i.e., $(1/n, 1/n, 1/n, \ldots, 1/n)$ if n is odd and 0 otherwise.

(b) The vectors in the orthogonal complement satisfy

$$x_1 + x_2 + x_3 + \cdots = 0$$

$$x_1 - x_2 + x_3 - x_4 + \cdots = 0$$

and this system can be row-reduced into the simpler system

$$x_1 + x_3 + x_5 + \cdots = 0$$

$$x_2 + x_4 + x_6 + \cdots = 0$$

so the vectors in the orthogonal complement of x and y are of the form

$$\begin{pmatrix} -x_3 - x_5 - x_7 - \cdots \\ -x_4 - x_6 - x_8 - \cdots \\ x_3 \\ x_4 \\ x_5 \\ \vdots \end{pmatrix}$$

so a basis consists of

$$\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}$$

and

$$\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ \vdots \end{pmatrix}$$

taken together.

12. The plane orthogonal to $(1, 0, 1)$ is $x + z = 0$. The plane parallel to this which passes through $(0, 1, 1)$ is the plane $x + z = 1$.

13. (updated 8/14) Yes:

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

14. removed

15. Since $u(x,t)$ is periodic in x with period 2π, you can write

$$u(x,t) = \sum_{n=-\infty}^{\infty} a_n(t)e^{inx}$$

Plugging this into the differential equation gives

$$ia_n'(t) = -n^2 a_n(t)$$

which has solutions

$$a_n(t) = a_n(0)e^{in^2 t}.$$

If $u(x,0) = \sum_{n=-\infty}^{\infty} b_n e^{inx}$, then

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx} = \frac{1}{2\pi}\left.\frac{e^{-inx}}{-in}\right|_{x=0}^{\pi} = \frac{1}{2\pi}\frac{(-1)^n - 1}{-in}$$

$$\Rightarrow u(x,t) = \frac{1}{2} + \sum_{n\neq 0}\frac{1}{2\pi}\left(\frac{(-1)^n - 1}{-in}\right)e^{in^2 t}e^{inx}$$

16. If

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$$

then if $n \neq 0$,

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx$$

$$= \frac{1}{2\pi}\int_0^{\pi} 2xe^{inx}dx = \frac{2}{2\pi}\left(\left.\frac{xe^{-inx}}{-in}\right|_0^{\pi} - \left.\frac{e^{-inx}}{(-in)^2}\right|_0^{\pi}\right)$$

$$= \frac{1}{\pi n}\left(i\pi(-1)^n + \left(\frac{(-1)^n - 1}{n}\right)\right)$$

If $n = 0$, then

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx = \frac{\pi}{2}$$

so

$$f(x) = \frac{\pi}{2} + \sum_{n\neq 0}\frac{1}{\pi n}\left(i\pi(-1)^n + \left(\frac{(-1)^n - 1}{n}\right)\right)e^{inx}$$