Math 54 Final Practice 2

1. Solve the following system of linear equations:

$$\begin{cases} x + 2y = 3 \\ x - y = 0 \end{cases}$$

2. Write down bases for the image and the kernel of

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

3. Compute

$$\begin{pmatrix} -13 & -10 \\ 15 & 12 \end{pmatrix}^{100}$$

(you can leave your answer in terms of powers like 5^{100})

4. Give an example of matrices A and B such that A and B are invertible but such that $A + B$ is not invertible.

5. Are there finitely many or infinitely many 3×3 matrices all of whose eigenvalues are 1? Explain.

6. What are the side lengths of the triangle in $\mathbb{R}^3$ with vertices at $(2, 0, 1)$, $(3, 2, 2)$, and $(3, -1, -1)$.

7. Let P be the plane in $\mathbb{R}^4$ given by the equations

$$\begin{cases} x + y + z = 0 \\ y + z + w = 0 \end{cases}$$

Write down an orthonormal basis of P. Also write down an orthonormal basis of $P^{\perp}$.

8. For what real values of a are the vectors

$$\begin{pmatrix} a \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ a \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ a \end{pmatrix}$$

linearly independent? For what values of a do they span $\mathbb{R}^3$?

9. Let V be the plane in $\mathbb{R}^4$ given by the equations

$$\begin{cases} x + y = 0 \\ z + w = 0 \end{cases}$$

What point on V is closest to the point $(1, 0, 0, 0)$?

10. What is the inverse of

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}?$$

11. Let $x \in \mathbb{R}^n$ be the vector $(1, -1, 1, -1, 1, -1, 1, \ldots, \pm 1)$ (its components alternate between 1 and -1. Let $y \in \mathbb{R}^n$ be the vector $(1, 1, 1, \ldots, 1)$ (all of its components are 1).

    (a) What is the orthogonal projection of x onto the span of y?

    (b) Write down a basis for the orthogonal complement of the span of x and y.

12. Write down an equation of the form $ax + by + cz = d$ for the plane in $\mathbb{R}^3$ that passes through the point $(0, 1, 1)$ and is orthogonal to the line

$$t \mapsto t \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix}$$

13. Does there exist an $m \times n$ matrix ($m \geq 2$ and $n \geq 2$) whose columns are orthogonal but whose rows are not orthogonal? If so, give an example. If it is not possible, explain.

14. (Update 8/13: a typo in ths problem made is messier than I intended. I'm leaving it up here, but something with a solution this messy would not appear on the final.)

    Solve the following differential equation with initial conditions:

$$\frac{d^2 f}{dx^2} + 4f = x^3 + 1$$

$$f(0) = 2, \;\; f'(0) = -1$$

15. Write down a solution to the equation

$$i\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

    with initial condition

$$u(x, 0) = \begin{cases} 1 & \text{if } 2\pi k \leq x < 2\pi k + \pi \text{ for all } k \\ 0 & \text{if } 2\pi k + \pi \leq x < 2\pi k + 2\pi \text{ for all } k \end{cases}$$

    Assume that u is periodic with period 2π in the x-coordinate: $u(x + 2\pi, t) = u(x, t)$.

16. Compute the Fourier series of the 2π-periodic function f defined on the interval $[0, 2\pi)$ by

$$f(x) = \begin{cases} 2x & 0 \le x < \pi \\ 0 & \pi \le x < 2\pi \end{cases}$$

and extended to all of $\mathbb{R}$ by translating by multiples of 2π.