# Math 54 Final Practice 1 Solutions

1. Let $P$ be the plane $x+y+z=0$. A vector $(a,b,c)$ in $P^{\perp}$ must be such that $(a,b,c)\cdot(x,y,z)=0$ for all vectors $(x,y,z)$ in $P$. Therefore you want a vector $(a,b,c)$ such that $ax+by+cz=0$ if $x+y+z+0$. One such vector is $(1,1,1)$. Since $\dim(P)=2$ and $\dim\mathbb{R}^3=3$, then $\dim P^{\perp}=1$ so $(1,1,1)$ is a basis of $P^{\perp}$.

2. A point $(x,y,z)$ in $L$ satisfies

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t+2 \\ -t+1 \\ t \end{pmatrix}$$

for some $t$. A point $(x,y,z)$ in $P$ satisfies $-x+y+3z=8$. A point in both $P$ and $L$ satisfies both of these conditions, so $-(t+2)+(-t+1)+3z=8$. This implies that $t=9$. Therefore

$$\begin{pmatrix} 11 \\ -8 \\ 9 \end{pmatrix}$$

is the intersection of $L$ and $P$.

3. Solving $Ax=y$ for $x$ involves row reducing

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

so the only solution is $(-1,2,-1)$.

4. There should be six vectors in the basis of $V$. For points in $V$ you can write $x_1$ and $x_2$ in terms of the other variables. Specifically, the points in $V$ are the points of the form

$$\begin{pmatrix} -x_3-x_5-x_7 \\ -x_4-x_6-x_8 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{pmatrix}$$

$$= x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_6 \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_7 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_8 \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

so a basis of $V$ is

$$\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

5. (updated 8/14)

$$\det \begin{pmatrix} 9-\lambda & -6 & -10 \\ 0 & 1-\lambda & 0 \\ 8 & -6 & -9-\lambda \end{pmatrix} = -(\lambda-1)^2(\lambda+1)$$

Therefore the eigenvalues are 1 and $-1$. The eigenvectors with eigenvalue 1 are the nonzero vectors in

$$\ker \begin{pmatrix} 9-1 & -6 & -10 \\ 0 & 1-1 & 0 \\ 8 & -6 & -9-1 \end{pmatrix} = \begin{pmatrix} 8 & -6 & -10 \\ 0 & 0 & 0 \\ 8 & -6 & -10 \end{pmatrix}$$

and these are the vectors $(x, y, z)$ such that $8x - 6y - 10z = 0$. A basis for this plane is $(3, 4, 0)$ and $(0, 5, -3)$. The eigenvectors with eigenvalue $-1$ are the nonzero vectors in

$$\ker \begin{pmatrix} 9-(-1) & -6 & -10 \\ 0 & 1-(-1) & 0 \\ 8 & -6 & -9-(-1) \end{pmatrix} = \begin{pmatrix} 10 & -6 & -10 \\ 0 & 2 & 0 \\ 8 & -6 & -8 \end{pmatrix}$$

and $(1, 0, 1)$ is one such vector (you can find this by row-reducing, for example). Then

$$\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

is an eigenbasis (it is not the only eigenbasis).

6. You want to find the times $t$ when these two vectors point along the same direction or opposite each other. Therefore it's enough to find the times $t$ when their normalizations

$$\begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}, \begin{pmatrix} \sin(t) \\ -\cos(t) \end{pmatrix}$$

are equal or opposite. Therefore you want to find the times $t$ when

$$\begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix} = \pm \begin{pmatrix} \sin(t) \\ -\cos(t) \end{pmatrix}$$

and these times are when $\cos(t) = 0$ or $\sin(t) = 0$. These are $t = \frac{\pi n}{2}$ where $n$ is any integer.

7. (no question on the final is going to be this long) Much like the Fibonacci problem we tried in class, the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{pmatrix}$$

is such that

$$A \begin{pmatrix} a_{n-3} \\ a_{n-2} \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} a_{n-2} \\ a_{n-1} \\ 2a_{n-1} + a_{n-2} - 2a_{n-3} \end{pmatrix} = \begin{pmatrix} a_{n-2} \\ a_{n-1} \\ a_n \end{pmatrix}$$

Iterating this, one sees that

$$A^{n-2} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_{n-2} \\ a_{n-1} \\ a_n \end{pmatrix}$$

Therefore to compute $a_n$ one must compute the 3rd component of

$$A^{n-2} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = A^{n-2} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

To find this one must compute

$$A^{n-2}$$

and to compute that one must write

$$A = PDP^{-1}$$

where $D$ is diagonal, so that $A^{n-2} = PD^{n-2}P^{-1}$. To do this, $P$ is the matrix whose columns are the eigenvectors of $A$ and $D$ is the matrix whose diagonal entries are the corresponding eigenvalues.

Since

$$\det \begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -2 & 1 & 2-\lambda \end{pmatrix} = -\lambda^3 + 2\lambda^2 + \lambda - 2 = (\lambda - 2)(-\lambda^2 + 1)$$

the eigenvalues of $A$ are 2, 1, and $-1$. Therefore

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

An eigenvector with eigenvalue 2 is a nonzero vector in the kernel of

$$\begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ -2 & 1 & 0 \end{pmatrix}$$

of which one is

$$\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

and an eigenvector with eigenvalue 1 is a nonzero vector in the kernel of

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -2 & 1 & 1 \end{pmatrix}$$

of which one is

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

and an eigenvector with eigenvalue $-1$ is a nonzero vector in the kernel of

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -2 & 1 & 3 \end{pmatrix}$$

of which one is

$$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Therefore $P$ can be taken to be

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 4 & 1 & 1 \end{pmatrix}$$

$P^{-1}$ can be computed by row-reducing

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 1 & 0 \\ 4 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/3 & 0 & 1/3 \\ 0 & 1 & 0 & 1 & 1/2 & -1/2 \\ 0 & 0 & 1 & 1/3 & -1/2 & 1/6 \end{array}\right)$$

so that

$$P^{-1} = \frac{1}{6}\begin{pmatrix} -2 & 0 & 2 \\ 6 & 3 & -3 \\ 2 & -3 & 1 \end{pmatrix}$$

and

$$A^{n-2}\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 4 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2^{n-2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (-1)^{n-2} \end{pmatrix}\frac{1}{6}\begin{pmatrix} -2 & 0 & 2 \\ 6 & 3 & -3 \\ 2 & -3 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

and the third component of this is

$$\frac{16 \cdot 2^{n-2} - 3 - (-1)^{n-2}}{6}$$

8. The image is the span of the columns. Since

$$\det\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = 1 \neq 0$$

the first three columns are linearly independent. Since you can't have three linearly independent vectors in a 0-, 1-, or 2-dimensional subspace of $\mathbb{R}^3$ then these three columns span all of $\mathbb{R}^3$. Hence the columns of the original $3 \times 5$ matrix span all of $\mathbb{R}^3$. Therefore the dimension of the image is 3 and the dimension of the kernel must be 2.

9. There are lots of examples. Perhaps the easiest are diagonal matrices. The standard basis is an eigenbasis for the standard basis, and it's also orthonormal. Therefore the eigendecomposition $A = PDP^{-1}$ is the same as the singular decomposition $A = U\Sigma V^T$ where $U = V = P$ and $\Sigma = D$.

10. The vectors $(0, 1, 0)$ and $(1, 0, 1)$ are already orthogonal, so applying Gram-Schmidt gives $u_1 = (0, 1, 0)$ and $u_2 = (1/\sqrt{2}, 0, 1/\sqrt{2})$. Therefore the orthogonal projection of $(1, 1, 2)$ onto the plane spanned by $u_1$ and $u_2$ is

$$\left(\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \cdot u_1\right) u_1 + \left(\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \cdot u_2\right) u_2 = \begin{pmatrix} 3/2 \\ 1 \\ 3/2 \end{pmatrix}$$

11. A vector in $V_1$ can be written

$$a\begin{pmatrix}1\\0\\1\\0\\0\end{pmatrix}+b\begin{pmatrix}0\\1\\1\\0\\1\end{pmatrix}$$

for some $a$ and $b$ and a vector in $V_2$ can be written

$$c\begin{pmatrix}0\\1\\0\\1\\0\end{pmatrix}+d\begin{pmatrix}1\\1\\1\\0\\0\end{pmatrix}+e\begin{pmatrix}1\\0\\0\\1\\1\end{pmatrix}$$

for some $c$, $d$, and $e$, so a vector in both satisfies

$$a\begin{pmatrix}1\\0\\1\\0\\0\end{pmatrix}+b\begin{pmatrix}0\\1\\1\\0\\1\end{pmatrix}=c\begin{pmatrix}0\\1\\0\\1\\0\end{pmatrix}+d\begin{pmatrix}1\\1\\1\\0\\0\end{pmatrix}+e\begin{pmatrix}1\\0\\0\\1\\1\end{pmatrix}$$

i.e.,

$$\begin{cases}a=d+e\\b=c+d\\a+b=d\\c+e=0\\b=e\end{cases}$$

and from these equations it's not hard to see that the only solution of $a=b=c=d=e=0$. For example, the last two equations say that $b=e$ and $c=-e$. Then the first and third say that $a+e=d$ and $a-e=e$ so that $a=d$ and $e=0$. Then $b=c=0$. The second equation then implies that $a=d=0$. Therefore the intersection of $V_1$ and $V_2$ is the point $(0,0,0,0,0)$.

12. removed

13. (a) linearly dependent
    (b) linearly dependent
    (c) linearly independent

14. To find the homogeneous solution, guess $f(x)=e^{rx}$ so that $r^2+2r-3=0$. Therefore $r=-3$ or $r=1$. The homogeneous solution is

$$f(x)=C_1e^{-3x}+C_2e^{x}$$

For the paticular solution, guess $f(x) = Ax + B$ so that

$$2(Ax+B)' - 3(Ax+B) = x+1 \Rightarrow 2A - 3Ax - 3B = x+1$$

Equating coefficients on each side implies that $A = -\frac{1}{3}$ and $B = -\frac{5}{9}$ so the general solution is

$$-\frac{1}{3}x - \frac{5}{9} + C_1 e^{-3x} + C_2 e^{x}$$

$$f(0) = 1 \Rightarrow -\frac{5}{9} + C_1 + C_2 = 1$$

$$f'(0) = 1 \Rightarrow -\frac{1}{3} - 3C_1 + C_2 = 1$$

$$\Rightarrow C_1 = \frac{1}{18},\ C_2 = \frac{3}{2}$$

Therefore the solution is

$$-\frac{1}{3}x - \frac{5}{9} + \frac{1}{18}e^{-3x} + \frac{3}{2}e^{x}$$

15. Write $u(x,t) = \sum_{n=-\infty}^{\infty} a_n(t)e^{inx}$. Since $u(x,0) = 2e^{2ix} + 2e^{-2ix}$ then

$$a_n(0) = \begin{cases} 2 & \text{if } n = \pm 2 \\ 0 & \text{otherwise} \end{cases}.$$

The coefficients $a_n(t)$ satisfy

$$a_n(t) = -n^2 a_n(0)$$

so

$$u(x,t) = 2e^{2ix}e^{-4t} + 2e^{-2ix}e^{-4t}$$

16. On the interval $[0, 2\pi)$:

$$f(x) = \frac{e^{ix/2} + e^{-ix/2}}{2}$$

If $f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$ then

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{e^{ix/2} + e^{-ix/2}}{2}\right) e^{-inx} dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\left(\frac{1}{2}-n\right)x} + e^{-i\left(\frac{1}{2}+n\right)x}}{2} dx$$

$$= \frac{1}{4\pi}\left[\left.\frac{e^{i\left(\frac{1}{2}-n\right)x}}{i\left(\frac{1}{2}-n\right)}\right|_0^{2\pi} + \left.\frac{e^{-i\left(\frac{1}{2}+n\right)x}}{-i\left(\frac{1}{2}+n\right)}\right|_0^{2\pi}\right]$$

$$= \frac{1}{2\pi i}\left[\frac{1}{\frac{1}{2}+n} - \frac{1}{\frac{1}{2}-n}\right]$$

$$= \frac{1}{2\pi i}\frac{-2n}{\frac{1}{4}-n^2}$$

so

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi i}\frac{-2n}{\frac{1}{4}-n^2}e^{inx}$$