Math 54 Final Practice 1

1. What is a basis of the orthogonal complement of $x + y + z = 0$?

2. Let L be the line
$$t \mapsto t \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$
Let P be the plane given by $-x + y + 3z = 8$. What is the intersection of P and L?

3. Let
$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$
Let
$$y = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$$
Find all vectors x such that $Ax = y$.

4. Let V be the six-dimensional subset of $\mathbb{R}^8$ given by the equations
$$x_1 + x_3 + x_5 + x_7 = 0$$
$$x_2 + x_4 + x_6 + x_8 = 0$$
Write down a basis of V.

5. Find an eigenbasis of
$$\begin{pmatrix} 9 & -6 & -10 \\ 0 & 1 & 0 \\ 8 & -6 & -9 \end{pmatrix}$$

6. The following two vectors vary with time:
$$v_1(t) = \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}$$
$$v_2(t) = \begin{pmatrix} e^t \sin(t) \\ -e^t \cos(t) \end{pmatrix}$$
For what values of t are they linearly dependent?

7. Let $a_0, a_1, a_2, a_3, \ldots$ be a sequence such that $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ and $a_0 = 0$, $a_1 = 1$, and $a_2 = 2$. Write down a formula for a_n. (hint: if you have a polynomial like $a\lambda^3 + b\lambda^2 + a\lambda + b$, you can factor out $a\lambda + b$)

8. What are the dimensions of the image and the kernel of

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}?$$

9. Give an example of a matrix A and an eigenbasis for A such that the vectors in the eigenbasis form singular vectors for A.

10. What is the orthogonal projection of $(1, 1, 2)$ onto the plane spanned by $(0, 1, 0)$ and $(1, 0, 1)$?

11. Let V_1 be the linear subspace of $\mathbb{R}^5$ spanned by $(1, 0, 1, 0, 0)$ and $(0, 1, 1, 0, 1)$. Let V_2 be the linear subspace of $\mathbb{R}^5$ spanned by $(0, 1, 0, 1, 0)$, $(1, 1, 1, 0, 0)$ and $(1, 0, 0, 1, 1)$. What is the intersection of V_1 and V_2?

12. removed

13. For each of the following, determine if each of the sets of vectors are linearly dependent or independent:

   (a)

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

   (b)

$$\begin{pmatrix} 49 \\ 12 \\ 37 \end{pmatrix}, \begin{pmatrix} -62 \\ -16 \\ -46 \end{pmatrix}, \begin{pmatrix} -44 \\ -12 \\ -32 \end{pmatrix}$$

   (c)

$$\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 2 \\ -2 \end{pmatrix}$$

14. Solve the following differential equation with initial conditions:

$$\frac{d^2 f}{dx^2} + 2\frac{df}{dx} - 3f = x + 1$$

$$f(0) = 1, \ f'(0) = 1$$

15. Recall that the heat equation is the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$u(x, 0) = f(x)$$

for some initial function f. Suppose that function u satisfies the heat equation with $f(x) = 4\cos(2x)$ and that $u(x + 2\pi, t) = u(x, t)$ for each time t. Write down an explicit formula for $u(x, t)$. (Hint: write cosine as a sum of two complex exponential functions).

16. Compute the Fourier series of the 2π-periodic function f defined on the interval $[0, 2\pi)$ by

$$f(x) = \cos(x/2)$$

and extended to all of $\mathbb{R}$ by translating by multiples of 2π.