The Fibonacci numbers

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ldots$$

are obtained by starting with $a_1 = 1$ and $a_2 = 1$, then recursively defining $a_n = a_{n-1} + a_{n-2}$. It will help some of the later calculations to set $a_0 = 0$. This is consistent since the recursion then defines a_2 to be $0 + 1 = 1$.

The recurrence relation for the Fibonacci sequence can be encoded in a matrix:

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a_{n-2} \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} a_{n-1} \\ a_{n-1} + a_{n-2} \end{pmatrix} = \begin{pmatrix} a_{n-1} \\ a_n \end{pmatrix}$$

In particular, one can compute the Fibonacci numbers by applying this matrix to $(a_0, a_1) = (0, 1)$:

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^4 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

Thus if a_n is the nth Fibonacci number, then

$$\begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

One can compute a formula for $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n$ in terms of its eigenvalues.

Its eigenvalues are the roots of

$$\det \begin{pmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} = 0$$

i.e.,

$$\lambda^2 - \lambda - 1 = 0$$

so $\lambda = \frac{1\pm\sqrt{5}}{2}$. Let $\phi = \frac{1+\sqrt{5}}{2}$. The other root turns out to be $-1/\phi$, since $-2/(1+\sqrt{5}) = (1-\sqrt{5})/2$.

Therefore

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = P \begin{pmatrix} \phi & 0 \\ 0 & -\frac{1}{\phi} \end{pmatrix} P^{-1}$$

where P is a matrix whose columns are eigenvectors of $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. To determine P one has to compute eigenvectors. These are elements in kernel of

$$\begin{pmatrix} -\phi & 1 \\ 1 & 1-\phi \end{pmatrix}$$

and

$$\begin{pmatrix} 1/\phi & 1 \\ 1 & 1+\frac{1}{\phi} \end{pmatrix}$$

respectively. Since $\phi^2 - \phi - 1 = 0$, then $1 - \phi = -\frac{1}{\phi}$. Therefore the first matrix is

$$\begin{pmatrix} -\phi & 1 \\ 1 & -\frac{1}{\phi} \end{pmatrix}$$

and so $(1, \phi)$ is a vector in the kernel. Similarly, $1 + \frac{1}{\phi} = -\phi$, so the second matrix is

$$\begin{pmatrix} 1/\phi & 1 \\ 1 & -\phi \end{pmatrix}$$

and $(1, -1/\phi)$ is a vector in the kernel. Therefore an eigenbasis is

$$\begin{pmatrix} 1 \\ \phi \end{pmatrix}, \begin{pmatrix} 1 \\ -\frac{1}{\phi} \end{pmatrix}$$

so

$$P = \begin{pmatrix} 1 & 1 \\ \phi & -\frac{1}{\phi} \end{pmatrix}$$

and

$$P^{-1} = \frac{1}{-\phi - \frac{1}{\phi}} \begin{pmatrix} -\frac{1}{\phi} & -1 \\ -\phi & 1 \end{pmatrix}$$

In fact, $\phi + \frac{1}{\phi} = \sqrt{5}$, so:

$$P^{-1} = \frac{1}{-\sqrt{5}} \begin{pmatrix} -\frac{1}{\phi} & -1 \\ -\phi & 1 \end{pmatrix}$$

then

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n = P \begin{pmatrix} \phi^n & 0 \\ 0 & \left(-\frac{1}{\phi}\right)^n \end{pmatrix} P^{-1}$$

which, after a lot of multiplication, is

$$\frac{1}{\sqrt{5}} \begin{pmatrix} \phi^{n-1} - \left(-\frac{1}{\phi}\right)^{n-1} & \phi^n - \left(-\frac{1}{\phi}\right)^n \\ \phi^n - \left(-\frac{1}{\phi}\right)^n & \phi^{n+1} - \left(-\frac{1}{\phi}\right)^{n+1} \end{pmatrix}$$

Therefore

$$\begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} \phi^{n-1} - \left(-\frac{1}{\phi}\right)^{n-1} & \phi^n - \left(-\frac{1}{\phi}\right)^n \\ \phi^n - \left(-\frac{1}{\phi}\right)^n & \phi^{n+1} - \left(-\frac{1}{\phi}\right)^{n+1} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

so that

$$a_n = \frac{\phi^n - \left(-\frac{1}{\phi}\right)^n}{\sqrt{5}}.$$

Note that this is

$$\frac{\phi^n}{\sqrt{5}} - \frac{(-1/\phi)^n}{\sqrt{5}}$$

the second term of which has absolute value less than 1/2. Therefore another way to compute a_n is to compute the integer closest to $\frac{\phi^n}{\sqrt{5}}$.