The number i is a number such that $i^2 = -1$. Real numbers always square to positive numbers, so i has to be combined with the real numbers "by hand". Suppose you combined the real numbers with this new number i. Then you'd need to allow i to be added to real numbers. Hence you should also allow numbers like $5 + i$. You should also allow yourself to multiply it by real numbers, so you should also allows numbers like $2i$, which of course is also equal to $i + i$. Note that $i^2 = -1$ so $i^3 = -i$, $i^4 = 1$, $i^5 = i$, $i^6 = -1$ etc. The pattern repeats every four powers, so $i^{4n} = 1$.

The resulting set of numbers is called the set of "complex numbers". These are numbers of the form $a+bi$ where a and b are real numbers. You manipulate complex numbers as if i is a variable like "x" or "y":

$$2(5 + i) = 10 + 2i$$

$$0i = 0$$

$$(5 + 2i)(1 - i) = 5 + 2i - 5i - 2i^2 = 5 + 2i - 5i - 2(-1) = 7 - 3i$$

Just as $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers.

Because complex number depend on two real numbers, you can represent $a + bi$ as the pair (a, b) and draw it on the cartesian plane. For example, $5+2i$ sits at the point $(5, 2)$ and the real numbers form the x-axis. The y-axis consists of the complex numbers of the form ai where a is real. The numbers on the y-axis are called "imaginary" numbers, though that's just a name. They're no less figments of our imagination than, say, negative numbers.

There's a notion of "absolute value" for real numbers. Namely, $|x|$ is the distance between x and 0. Similarly let $|a + bi|$ be the distance between $a + bi$ and 0. By the pythagorean theorem, this is $\sqrt{a^2 + b^2}$. If $z = a + bi$, let $\overline{z} = a - bi$. $\overline{z}$ is called the "complex conjugate" of z. It is not hard to check that, if z and w are complex numbers, $\overline{zw} = \overline{z}\overline{w}$ and $|z| = \sqrt{z\overline{z}}$. Therefore $|zw| = |z||w|$ and $z\overline{z} = |z|^2$ is always real.

You can divide complex numbers as well as add, subtract, and multiply as well:

$$\frac{1 + i}{2 - i}$$

To get this in the form $a + bi$, multiply top and bottom by the complex conjugate of the denominator

$$\frac{1 + i}{2 - i} \cdot \frac{2 + i}{2 + i} = \frac{(1 + i)(2 + i)}{5} = \frac{1 + 3i}{5} = \frac{1}{5} + \frac{3}{5}i$$

The complex numbers z with $|z| = 1$ play a special role. These are called the unit complex numbers. Since these lie on the unit circle, they can written as $\cos\theta + i\sin\theta$, where θ is the angle measured counterclockwise from the positive x-axis. In fact, you can write unit complex numbers in a more efficient way. The power series expressions for e^x, $\cos(x)$, and $\sin(x)$ are all very similar:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

It would be true that $e^x = \cos(x) + \sin(x)$ if it weren't for those pesky minus signs. This can be remedied by placing i in the appropriate places (note that $i^3 = -i$, $i^4 = 1$, $i^5 = i$, etc):

$$e^{ix} = 1 + ix - \frac{x^2}{2} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} + \cdots$$

so

$$e^{ix} = \cos(x) + i\sin(x)$$

This miraculous formula says that $\cos(x)$ and $\sin(x)$ can be packaged together using the exponential function.

Therefore any unit complex number is $e^{i\theta}$ for some angle θ. In fact, if z is a nonzero complex number then $|z/|z|| = 1$, so $z/|z| = e^{i\theta}$ so $z = |z|e^{i\theta}$ for some angle θ. For example

$$1 + i = \sqrt{2}(\cos(\pi/4) + i\sin(\pi/4)) = \sqrt{2}e^{i\pi/4}.$$

So all complex numbers can be represented as sums $a+bi$. All nonzero complex numbers can be alternatively represented as products $re^{i\theta}$.

Multiplying a complex number by a real number scales it by that real number: $5(a + bi) = 5a + 5bi$. Since $e^{x+y} = e^x e^y$, it follows that $(re^{i\theta})e^{i\varphi} = re^{i(\theta+\varphi)}$. Thus multiplying a complex number by a unit complex number $e^{i\varphi}$ rotates it counterclockwise by θ.

Incidentally, most trig identities follow easily from the equation $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. For example,

$$\cos(\theta+\varphi)+i\sin(\theta+\varphi) = e^{i(\theta+\varphi)} = e^{i\theta}e^{i\phi} = (\cos(\theta)+i\sin(\theta))(\cos(\varphi)+i\sin(\varphi))$$
$$= \cos(\theta)\cos(\varphi) + i\sin(\theta)\cos(\varphi) + i\cos(\theta)\sin(\varphi) - \sin(\theta)\sin(\varphi)$$
$$= (\cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi)) + i(\sin(\theta)\cos(\varphi) + \cos(\theta)\sin(\varphi))$$

so

$$\cos(\theta + \varphi) = \cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi)$$
$$\sin(\theta + \varphi) = \sin(\theta)\cos(\varphi) + \cos(\theta)\sin(\varphi)$$

The quadratic equation

$$ax^2 + bx + c = 0$$

has two solutions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $b^2 - 4ac$ is negative, then these two solutions are not real, but instead involve i, for example

$$x^2 + x + 1$$

has solutions

$$x = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{(-1)3}}{2} = \frac{-1 \pm \sqrt{-1}\sqrt{3}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

In fact all polynomial equations, in all degrees, have solutions over the complex numbers. And some polynomial equations have more solutions than just the real solutions you're already familiar with.

For example, the equation $x^3 - 1 = 0$ has one real solution, namely 1.

But it has two other complex solutions. In fact it is not hard to see that these solutions must be the points 1/3 of the way around the unit circle and 2/3 of the way around the unit circle. These are the points $e^{2\pi i/3}$ and $e^{4\pi i/3}$. For example $(e^{2\pi i/3})^3 = e^{2\pi i/3}e^{2\pi i/3}e^{2\pi i/3} = e^{2\pi i/3+2\pi i/3+2\pi i/3} = e^{2\pi i}$. And $e^{2\pi i}$ is the point an angle of 2π around the unit the circle, so $e^{2\pi i} = 1$. One could also obtain these other solutions by factoring

$$x^3 - 1 = (x - 1)(x^2 + x + 1)$$

so if $x^3 - 1 = 0$ then $x = 1$ or $x^2 + x + 1 = 0$, i.e.,

$$x = \frac{-1 \pm i\sqrt{3}}{2}$$

$-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ are the two points $e^{2\pi i/3}$ and $e^{4\pi i/3}$ written in the form $a + bi$.