If v_1, v_2 is a basis of $\mathbb{R}^2$ then there is one way to write every vector in $\mathbb{R}^2$ as a linear combination of v_1 and v_2. For example, if

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

then

$$\begin{pmatrix} 5 \\ 3 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 5v_1 + 3v_2.$$

Or if

$$w_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; w_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

is another basis then

$$\begin{pmatrix} 5 \\ 3 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 4w_1 - w_2$$

One can determine these coefficients by solving

$$\begin{pmatrix} 5 \\ 3 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

for a_1 and a_2.

Exercise. Consider the basis

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \; u_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, u_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

of $\mathbb{R}^3$. Write the vector

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

as a linear combination of these basis vectors.

Solution.

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

An important fact: a linear map is determined by what it does to a basis. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map and let $u_1, \ldots, u_n$ be a basis of $\mathbb{R}^n$. Let u be a vector in $\mathbb{R}^n$. there is a unique way to write

$$u = a_1 u_1 + \ldots + a_n u_n$$

Then

$$T(u) = a_1 T(u_1) + \ldots + a_n T(u_n)$$

so T is completely determined by the n vectors $T(u_1), \ldots, T(u_n)$.

For example given the two bases

$$v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \; v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

and

$$w_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \; w_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

there is a unique transformation of $\mathbb{R}^2$ that takes v_1 to w_1 and v_2 to w_2. This transformation is called a "change of basis transformation". Its inverse takes w_1 to v_1 and w_2 to v_2.

In general, given two bases $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_n)$ of $\mathbb{R}^n$, the change of basis transformation from $(v_1, \ldots, v_n)$ to $(w_1, \ldots, w_n)$ is the linear map that takes v_i to w_i for each $i \in \{1, \ldots, n\}$.

It can be a little tricky to write down matrices for change of basis transformations. The exception to this is when one of the bases is the "standard basis".

Let e_i be the vector which is all 0s except a 1 in the ith entry. For example, in $\mathbb{R}^2$,

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

or, in $\mathbb{R}^3$,

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \; e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \; e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The basis $(e_1, \ldots, e_n)$ is called the "standard basis" and the vectors in the standard basis are "standard basis vectors".

Note that if A is a matrix then Ae_i is the ith column of A. Therefore if $(v_1, \ldots, v_n)$ is a basis of $\mathbb{R}^n$ then the matrix whose columns are $v_1, \ldots, v_n$ gives the linear transformation that takes e_i to v_i.

For example,

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

takes the standard basis (e_1, e_2) to the basis

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

Exercise. Write down the matrix that takes the standard basis of $\mathbb{R}^3$ to the basis

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

Solution.

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

Exercise. Write down the matrix that takes the basis

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

to the standard basis.

Solution. This is just the inverse of the matrix from the previous exercise:

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & 0 \end{pmatrix}$$

Exercise. Write down the matrix that takes the basis

$$v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

to the basis

$$w_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, w_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

(Hint: first write down matrices that take the standard basis to each of these bases.)

Solution. The matrix that takes the standard basis to (v_1, v_2) is

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

and the matrix that takes the standard basis to (w_1, w_2) is

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Therefore the matrix that takes the basis (v_1, v_2) to (w_1, w_2) is the composition of the matrix that takes (v_1, v_2) to (e_1, e_2) followed by the transformation that takes (e_1, e_2) to (w_1, w_2):

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 5/3 & -4/3 \\ -4/3 & 5/3 \end{pmatrix}$$

Lay uses some idiosyncratic notation for changing bases. If (v_1, v_2) is a basis of $\mathbb{R}^2$, for example, then he writes

$$\big[w\big]_{(v_1,v_2)} = \begin{bmatrix} a \\ b \end{bmatrix}$$

to mean $w = av_1 + bv_2$. I do not use this notation.