Math 110 Midterm 2 (SOLUTIONS)

1. (a) With respect to an orthonormal basis, the matrix for the adjoint is the complex conjugate transpose. Therefore the matrix for the adjoint of T, with respect to the standard basis, is

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

(b) Since $(ST)^* = T^*S^*$, it follows that $(TT^*)^* = (T^*)^*T^* = TT^*$, so T^*T is self-adjoint.

2. Since T is normal and V is complex, T has an orthonormal eigenbasis. Let $(u_1, \ldots, u_n)$ be this eigenbasis and let λ_i be the eigenvalue for u_i. Write v in terms of this basis

$$v = a_1u_1 + \cdots + a_nu_n$$

so that

$$Tv = a_1\lambda_1u_1 + \cdots + a_n\lambda_nu_n$$
$$\|Tv\|^2 = |a_1|^2|\lambda_1|^2 + \cdots + |a_n|^2|\lambda_n|^2.$$

Since each $|\lambda_i| < 1$ for all i,

$$\|Tv\|^2 \leq |a_1|^2 + \cdots + |a_n|^2 = \|v\|^2$$

hence

$$\|Tv\| \leq \|v\|.$$

3. (a) Since V is a real inner product space,

$$\langle v, w\rangle = \langle w, v\rangle.$$

$$\|v + w\|^2 + \|v - w\|^2 = \langle v + w, v + w\rangle + \langle v - w, v - w\rangle$$
$$= \|v\|^2 + 2\langle v, w\rangle + \|w\|^2 + \|v\|^2 - 2\langle v, w\rangle + \|w\|^2 = 2(\|v\|^2 + \|w\|^2).$$

(b) Suppose for a contradiction that there is such an inner product. Then part (a) holds. Letting

$$v = \begin{pmatrix} a \\ b \end{pmatrix} \quad w = \begin{pmatrix} c \\ d \end{pmatrix}$$

then

$$\max(|a+c|, |b+d|)^2 + \max(|a-c|, |b-d|)^2 = 2(\max(a, b)^2 + \max(c, d)^2).$$

for all $a, b, c, d \in \mathbb{R}$. But this isn't always true. For example, set $a = 1$, $b = 0$, $c = 1$, $d = 1$. Then the left hand side is 5 and the right hand side is 4.

4. It is easy to see that the the three vectors are linearly independent, and thus form a basis of U. Therefore $\dim(U) = 3$ so that $\dim(U^\perp) = 1$. Note that $U^\perp$ is the set of vectors (a, b, c, d) such that

$$a + b = 0, \;\; a + d = 0, \;\; c + d = 0$$

i.e., the set of vectors of the form $(a, -a, a, -a)$. For example, $(1, -1, 1, -1)$ is a basis vector that can be normalized to

$$w = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}.$$

Let

$$v = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 4 \end{pmatrix}.$$

Then the projection of v to U is

$$v - (v \cdot w)w = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 4 \end{pmatrix} - \frac{1}{4}(1 - 2 + 1 - 4) \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 3 \end{pmatrix}.$$