Math 110 Midterm 1 (SOLUTIONS)

1. $S \circ T = 0 \Leftrightarrow S(T(v)) = 0\ \forall v \in V \Leftrightarrow T(v) \in \ker(S)\ \forall v \in V \Leftrightarrow \operatorname{im}(T) \subset \ker(S)$.

2. **Solution 1:** Let V be the vector space of polynomials of degree less than or equal to $n-1$. Since $\dim(V) = n$, the n linearly independent vectors

$$\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \cdots, \begin{pmatrix} x_1^{n-1} \\ x_2^{n-1} \\ \vdots \\ x_n^{n-1} \end{pmatrix}$$

form a basis. Therefore given $(y_1, \ldots, y_n) \in \mathbb{F}^n$ there are unique elements $a_i \in \mathbb{F}$ such that

$$a_0 \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} + a_1 \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \cdots + a_{n-1} \begin{pmatrix} x_1^{n-1} \\ x_2^{n-1} \\ \vdots \\ x_n^{n-1} \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}.$$

Set $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$. Then $p(x_i) = y_i$ for each i.

The uniqueness of the a_i show that p is unique. In more words: suppose that there were two polynomials p and q of degree less than or equal to n such that $p(x_i) = y_i$ and $q(x_i) = y_i$. Then $p(x_i) - q(x_i) = 0$ for all i. Write

$$p(x) - q(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_{n-1} x^{n-1}.$$

Then

$$b_0 \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} + b_1 \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \cdots + b_{n-1} \begin{pmatrix} x_1^{n-1} \\ x_2^{n-1} \\ \vdots \\ x_n^{n-1} \end{pmatrix} = 0.$$

By the linear independence assumption, all of the b_is must be 0, so $p = q$.

Solution 2: Build a matrix

$$M = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix}.$$

Let

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}.$$

An element in $\ker(M)$ provides a linear relation amongst the columns

$$Ma = a_0 \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} + a_1 \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \cdots + a_{n-1} \begin{pmatrix} x_1^{n-1} \\ x_2^{n-1} \\ \vdots \\ x_n^{n-1} \end{pmatrix}$$

where

$$a = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix}.$$

Therefore, by the linear independence assumption, M is injective. By rank-nullity, M must be surjective as well, so the equation

$$Ma = y$$

has a solution for some $a \in \mathbb{F}^n$. If $a = (a_0, a_1, \ldots, a_n)$ is this solution,

$$a_0 + a_1 x_i + a_2 x_i^2 + \cdots + a_{n-1} x_i^{n-1} = y_i$$

for all i.

Uniqueness is more or less the same proof as in Solution 1.

3. T is diagonalizable if and only if it has a basis of eigenvectors. I will show that all the eigenvectors of T lie on a single line and therefore cannot span $\mathbb{R}^2$. Let $\begin{pmatrix} a \\ b \end{pmatrix}$ be an eigenvector for T. Then

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}$$

so

$$\begin{pmatrix} 0 \\ a \end{pmatrix} = \begin{pmatrix} \lambda a \\ \lambda b \end{pmatrix}.$$

If $\lambda \neq 0$, then these two equations show that $a = 0$ and $b = a/\lambda = 0$. This cannot be the case since eigenvectors are nonzero. Therefore $\lambda = 0$, and so $a = 0$. Therefore all eigenvectors lie on the 1-dimensional subspace of vectors

$$\begin{pmatrix} 0 \\ b \end{pmatrix}.$$

4. The linear transformation T from the last question gives an example where $V \neq \ker(T) \oplus \operatorname{im}(T)$. Since $\ker(T) = \operatorname{Span}(e_2) = \operatorname{im}(T)$, then $\ker(T) \cap \operatorname{im}(T) \neq \{0\}$ so the sum $\ker(T) + \operatorname{im}(T)$ cannot be direct.