Math 110 Practice Midterm 2

1. Prove or disprove:
$$\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}$$
is a basis of $\mathbb{R}^2$.

2. (a) Let $T : V \to W$ be a linear map. Prove that $\dim(\mathrm{im}(T)) \leq \dim(W)$.

   (b) Let $T : V \to W$ be a linear map. Prove that $\dim(\mathrm{im}(T)) \leq \dim(V)$.

3. Let V be a vector space and U, U', W subspaces. Prove or disprove: if $V = U \oplus W$ and $V = U' \oplus W$ then $U = U'$.

4. Recall that the space of $m \times n$ matrices is a vector space with addition $(M+N)_{ij} := M_{ij} + N_{ij}$ and scalar multiplication $(cM)_{ij} = cM_{ij}$. If M is a matrix, its transpose $M^\top$ is the matrix defined by $M^\top_{ij} = M_{ji}$.

   An $n \times n$ matrix is symmetric if $M^\top = M$.

   (a) Prove that $n \times n$ symmetric matrices form a subspace of the space of $n \times n$ matrices.

   (b) Find a basis for the space of $n \times n$ symmetric matrices.

5. Find a linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T^2 \neq 0$ but $T^3 = 0$.