Math 110 Practice Midterm 1

1. Prove or disprove: there exists a vector space V with three subspaces U_1, U_2, U_3 such that $U_1 \cap U_2$, $U_2 \cap U_3$ and $U_3 \cap U_1$ are all nonzero but $U_1 \cap U_2 \cap U_3 = \{0\}$.

2. Let V be a vector space and let n be a positive integer. Show that the following conditions are equivalent
   - $\dim(V) \leq n$
   - Any collection of more than n vectors in V is linearly dependent.

3. Let V be a finite-dimensional vector space. Let W be a subspace of V. Prove that there exists a subspace U of V such that $V = U \oplus W$.

4. Let V be a finite-dimensional vector space and U a subspace. Define the set V/U to be the set of subsets of V of the form $v + U$ for some $v \in V$. V/U is a vector space with addition

$$(v + U) + (w + U) := (v + w + U)\ , v, w \in V$$

   and scalar multiplication

$$c(v + U) := cv + U,\ c \in \mathbb{F},\ v \in V.$$

   (a) What is the element 0 (the additive identity) in V/U?

   (b) Prove that $\dim(V/U) = \dim(V) - \dim(U)$. (Hint: use a linear map $V \to V/U$.)