Math 110 Homework 6 (SOLUTIONS)

1. Suppose the matrix is $n \times n$. Doing the determinants explicitly for the cases $n = 1, 2, 3, 4$ suggests that the determinant of the $n \times n$ case is $n+1$. This can be proved using induction. Let d_n denote the determinant for the $n \times n$ case. Suppose that $d_k = k+1$ for $k \leq n-1$. We'd like to prove that $d_n = n+1$. Take the determinant of the $n\times$n matrix by expanding down the first column

$$d_n = 2d_{n-1} + \det \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \ddots & 0 & 0 \\ 0 & 0 & -1 & \ddots & -1 & 0 \\ \vdots & \vdots & \vdots & \ddots & 2 & -1 \\ 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}$$

Expanding down the first column of this second determinant then shows that

$$d_n = 2d_{n-1} - d_{n-2} = 2n - (n-1) = n+1.$$

2. (a) Each term in the monomial is M_{ab} where σ maps b to a. These terms can written as $M_{\sigma(i)i}$ or as $M_{j\sigma^{-1}(j)}$ so

$$\prod_{i=1}^{n} M_{\sigma(i)i} = \prod_{j=1}^{n} M_{j\sigma^{-1}(j)}.$$

(b) Summing over the set of $\sigma \in S_n$ is the same as summing over the set of σ^{-1} in S_n since each permutation is the inverse of some permutation.

(c)

$$\begin{aligned} \det(M^\top) &= \sum_\sigma \operatorname{sign}(\sigma) M^\top_{\sigma(1)1} \cdots M^\top_{\sigma(n)n} = \sum_\sigma \operatorname{sign}(\sigma) M_{1\sigma(1)} \cdots M_{n\sigma(n)} \\ &= \sum_\sigma \operatorname{sign}(\sigma) M_{\sigma^{-1}(1)1} \cdots M_{\sigma^{-1}(n)n} \\ &= \sum_{\sigma^{-1}} \operatorname{sign}(\sigma^{-1}) M_{\sigma(1)1} \cdots M_{\sigma(n)n} \\ &= \sum_{\sigma^{-1}} \operatorname{sign}(\sigma) M_{\sigma(1)1} \cdots M_{\sigma(n)n} \\ &= \sum_{\sigma} \operatorname{sign}(\sigma) M_{\sigma(1)1} \cdots M_{\sigma(n)n} \\ &= \det(M) \end{aligned}$$

where the third equality is part (a), the fourth equality is a change of variables σ to σ^{-1}, the fifth is the fact that $\operatorname{sign}(\sigma) = \operatorname{sign}(\sigma^{-1})$, and the sixth is part (b).

3. Since M has orthonormal columns, that means it sends the orthonormal basis $e_1, \dots, e_n$ to the orthonormal basis $Me_1, \dots, Me_n$. Therefore M is an isometry, so $M^{-1} = M^\top$. Therefore $\det(M) = \det(M^{-1})$. But $\det(M)\det(M^{-1}) = 1$, so $\det(M)^2 = 1$. Therefore $\det(M) = \pm 1$.

4. The characteristic polynomial is

$$\det \begin{pmatrix} t-6 & -6 & -1 \\ 1 & t+1 & 1 \\ 6 & 6 & t+1 \end{pmatrix}$$

$$\begin{aligned} &= (t-6)\det\begin{pmatrix} t+1 & 1 \\ 6 & t+1 \end{pmatrix} - \det\begin{pmatrix} -6 & -1 \\ 6 & t+1 \end{pmatrix} + 6\det\begin{pmatrix} -6 & -1 \\ t+1 & 1 \end{pmatrix} \\ &= (t-6)[(t+1)^2 - 6] - [-6(t+1) + 6] + 6[-6 + t + 1] \\ &= (t-6)(t^2 + 2t - 5) + 12t - 30 \\ &= t^3 - 4t^2 - 5t = t(t-5)(t+1) \end{aligned}$$

Therefore the eigenvalues are 0, 5, and -1.

5. let $\lambda_1, \dots, \lambda_n$ be the diagonal entries of the Jordan form for M. Then $\det(tI - M) = \prod_{i=1}^n (t - \lambda_i)$. If $\det(tI - M) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_0$ then a_k is $(-1)^k$ times the sum over all products of $n-k$ distinct elements in $\{\lambda_1, \dots, \lambda_n\}$. For example

$$t_n = 1$$

$$t_{n-1} = -(\lambda_1 + \cdots + \lambda_n)$$

$$t_{n-2} = \sum_{i<j} \lambda_i \lambda_j$$

$$t_{n-3} = -\sum_{i<j<k} \lambda_i \lambda_j \lambda_k$$

$$\vdots$$

$$t_0 = (-1)^n \lambda_1 \lambda_2 \cdots \lambda_n.$$