Math 110 Homework 5 (SOLUTIONS)

1. Extend $(e_1, \ldots, e_m)$ to a basis of all of V: $(e_1, \ldots, e_m, v_{m+1}, \ldots, v_n)$. Apply Gram-Schmidt to get an orthonormal basis of V (Gram-Schmidt will not change the first m vectors, since they're already orthonormal): $(e_1, \ldots, e_m, e_{m+1}, \ldots, e_n)$. Then

$$v = \sum_{i=1}^{n} \langle v, e_i \rangle e_i$$

and

$$\langle v, v \rangle = \sum_{i=1}^{n} |\langle v, e_i \rangle|^2.$$

This will equal

$$\sum_{i=1}^{m} |\langle v, e_i \rangle|^2$$

if and only if

$$\sum_{i=m+1}^{n} |\langle v, e_i \rangle|^2 = 0$$

which is the case if and only if

$$\langle v, e_i \rangle = 0$$

for all $i \geq m+1$ which is the case if and only if v is in the span of $e_1, \ldots, e_m$.

2.

3. Let $e_1, \ldots, e_n$ be the standard basis for $\mathbb{F}^n$. Write

$$T^* e_i = \sum_{j=1}^{n} a_j e_j$$

Then

$$\langle T^* e_i, e_k \rangle = a_k$$

so

$$a_k = \langle e_i, T e_k \rangle = \begin{cases} \langle e_i, e_{k+1} \rangle & k \neq n \\ 0 & k = n \end{cases}$$

so if a_k is 0 except when $k = i - 1$, in which case $a_k = 1$. Therefore

$$T^* e_i = e_{i-1},\ 2 \leq i \leq n$$

$$T^* e_1 = 0.$$

So

$$T^*(z_1, \ldots, z_n) = (z_2, z_3, \ldots, z_n, 0).$$

4. λ an eigenvalue of T if and only if $\ker(T-\lambda) \neq \{0\}$ if and only if $\dim(\ker(T-\lambda)) \neq 0$ if and only if $\dim(\text{im}((T-\lambda)) \neq \dim(V)$ if and only if $\dim(\ker(T-\lambda)^*)^\perp \neq \dim(V)$ if and only if $\dim(\ker((T-\lambda)^*) \neq 0$ if and only if $\ker((T-\lambda)^*) \neq \{0\}$ if and only if $\ker(T^* - \overline{\lambda}) \neq \{0\}$ if and only if $\overline{\lambda}$ is an eigenvalue for T^*.

5. Suppose T is self-adjoint. Then in class we proved all its eigenvalues are real. Suppose T that T is normal and all its eigenvalues are real. Let $(v_1, \ldots, v_n)$ be an eigenbasis for T with $Tv_i = \lambda_i v_i$. In class we showed that $T^* v_i = \overline{\lambda_i} v_i$. Since λ_i is real, $\overline{\lambda_i} = \lambda_i$ so $T^* v_i = \lambda_i v_i$. Since T^* is the same as T on the basis $(v_1, \ldots, v_n)$, then $T^* = T$.

6. One has to check all of the axioms of a vector space.

$$\langle v + w, u\rangle_T = \langle T(v+w), u\rangle = \langle Tv, u\rangle + Tw, u\rangle = \langle v, u\rangle_T + \langle w, u\rangle_T.$$

$$\langle cv, w\rangle_T = \langle T(cv), w\rangle = c\langle T(v), w\rangle = c\langle v, w\rangle_T.$$

$$\overline{\langle w, v\rangle_T} = \overline{\langle T(w), v\rangle} = \langle v, T(w)\rangle = \langle T(v), w\rangle = \langle v, w\rangle_T$$

(since T is self adjoint)

$$\langle v, v\rangle_T = \langle Tv, v\rangle \geq 0$$

(since T is positive semidefinite)

$$\langle v, v\rangle_T = 0 \Rightarrow \langle Tv, v\rangle_T = 0 \Rightarrow \langle T^{1/2} v, T^{1/2} v\rangle = 0 \Rightarrow T^{1/2} v = 0$$

where $T^{1/2}$ is the square root of T defined by

$$T^{1/2} v_i = \sqrt{\lambda_i} v_i$$

where $(v_1, \ldots, v_n)$ is an eigenbasis of T with eigenvalues λ_i. Since T is invertible each $\lambda_i > 0$, hence $T^{1/2}$ does not have zero as an eigenvalue. Hence $v = 0$.