Math 110 Homework 4 (SOLUTIONS)

1. (a) $\langle u+v, u-v\rangle = \langle u,u\rangle - \langle u,v\rangle + \langle v,u\rangle - \langle v,v\rangle$. Since this a real inner product space, $\langle u,v\rangle = \langle v,u\rangle$ so this is $\|u\|^2 - \|v\|^2$.
   (b) This follows from part (a)
   (c) Let $ABCD$ be the vertices of the rhombus. Let $B-A=u$ and $D-A$ be v. Then the diagonals are described by the vectors $D-B=u-v$ and $C-A=u+v$. Since the sides of a rhombus are all the same length, $\|u\|^2 = \|v\|^2$ so by part (a) $\langle u-v, u+v\rangle = 0$. Therefore the diagonals are orthogonal. In $\mathbb{R}^2$ with the dot product, orthogonal vectors are perpendicular (at right angles).

2. The condition $\|u\| \le \|u+av\|$ is the same as $\|u\|^2 \le \|u+av\|^2$ and

$$\|u+av\|^2 = \|u\|^2 + \langle u, av\rangle + \langle av, u\rangle + \|av\|^2$$

$$= \|u\|^2 + \overline{a}\langle u,v\rangle + a\overline{\langle u,v\rangle} + |a|^2\|v\|^2$$

$$= \|u\|^2 + 2\mathrm{Re}(a\overline{\langle u,v\rangle}) + |a|^2\|v\|^2.$$

Therefore $\|u\| \le \|u+av\|$ for all $a \in \mathbb{F}$ if and only if

$$0 \le |a|^2\|v\|^2 + 2\mathrm{Re}(a\overline{\langle u,v\rangle}).$$

If $\langle u,v\rangle = 0$ then of course this is true. The tricky part is the converse. Suppose that $\langle u,v\rangle \ne 0$ and that

$$0 \le |a|^2\|v\|^2 + 2\mathrm{Re}(a\overline{\langle u,v\rangle})$$

for all $a \in \mathbb{F}$. If $\|v\| = 0$, then $v=0$ and $\langle u,v\rangle$ holds trivially. Otherwise, set $a = -\frac{\langle u,v\rangle}{\|v\|}$. Then

$$0 \le |\langle u,v\rangle|^2 - 2|\langle u,v\rangle|^2 = -|\langle u,v\rangle|^2$$

which can only occur if $\langle u,v\rangle = 0$.

3. This is Cauchy-Schwarz in $\mathbb{R}^n$. Set

$$v = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad u = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

Then $\langle v,u\rangle^2 \le \|v\|^2\|u\|^2$ translates into

$$(x_1 + \cdots + x_n)^2 \le n(x_1^2 + \cdots + x_n^2).$$

4. Put the top of the triangle at the origin and let v and w be vectors representing the two sides of lengths a and b. Reflect the triangle across the side of length c to form a parallelogram. Then one sees that $\|v+w\| = 2d$. Since $\|v-w\| = c$, then

$$\|v\|^2 + \|w\|^2 = \frac{\|v+w\|^2 + \|v-w\|^2}{2}$$

implies

$$a^2 + b^2 = \frac{1}{2}c^2 + 2d^2.$$

5. (a) $v_1 = (3, 6, 0)$, $v_2 = (1, 2, 2)$.

$$u_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.$$

$$v_2 - (v_2 \cdot u_1)u_1 = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

$$u_2 = \frac{v_2 - (v_2 \cdot u_1)u_1}{\|v_2 - (v_2 \cdot u_1)u_1\|} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(b) A vector (a, b, c) is orthogonal to both v_1 and v_2 if

$$3a + 6b = 0 \text{ and } a + 2b + 2c = 0.$$

These two imply that $c = 0$ and $a = -2b$ so $W^\perp$ is the set of vectors of the form $(-2b, b, 0)$ for all $b \in \mathbb{R}$. A basis is the vector $(-2, 1, 0)$.

6. (a) Apply Gram-Schmidt to $v_1 = 1$, $v_2 = x$, and $v_3 = x^2$. Note that 1 is already normalized,

$$\int_0^1 1 \cdot 1 dx = 1.$$

Therefore $u_1 = 1$. Then

$$v_2 - \langle v_2, u_1 \rangle u_1 = x - \left(\int_0^1 x dx \right) 1 = x - \frac{1}{2}.$$

$$\|v_2 - \langle v_2, u_1 \rangle u_1\|^2 = \int_0^1 \left(x - \frac{1}{2} \right)^2 dx = \frac{1}{12}$$

so that

$$u_2 = \frac{v_2 - \langle v_2, u_1 \rangle u_1}{\|v_2 - \langle v_2, u_1 \rangle u_1\|} = \sqrt{12} \left(x - \frac{1}{2} \right).$$

$$v_3 - \langle v_3, u_2\rangle u_2 - \langle v_3, u_1\rangle u_1$$

is

$$x^2 - 12\left(\int_0^1 x^2\left(x-\frac{1}{2}\right)dx\right)\left(x-\frac{1}{2}\right) - \left(\int_0^1 x^2 dx\right)1 = x^2 - x + \frac{1}{6}.$$

and since

$$\int_0^1 \left(x^2 - x + \frac{1}{6}\right)^2 dx = \frac{1}{180}$$

so

$$u_3 = \sqrt{180}\left(x^2 - x + \frac{1}{6}\right).$$

Therefore the orthonormal basis is

$$1,\ \sqrt{12}\left(x-\frac{1}{2}\right),\ \sqrt{180}\left(x^2 - x + \frac{1}{6}\right).$$

(b) You could keep on applying Gram-Schmidt to $x^4, x^5, x^6, \ldots$ to get an infinite linearly independent subset of $W^\perp$. Therefore $W^\perp$ is infinite-dimensional.

(c)

$$P_W(x^3) = \langle x^3, u_1\rangle u_1 + \langle x^3, u_2\rangle u_2 + \langle x^3, u_3\rangle u_3$$

$$\langle x^3, u_1\rangle = \frac{1}{4}$$

$$\langle x^3, u_2\rangle = \frac{3\sqrt{12}}{40}$$

$$\langle x^3, u_3\rangle = \frac{\sqrt{180}}{120}.$$

so that

$$P_W(x^3) = \frac{1}{20} - \frac{3}{5}x + \frac{3}{2}x^2.$$