Math 110 Homework 3 (SOLUTIONS)

1. Let $w \in \mathrm{im}(S)$. Then $w = S(v)$ for some v. $T(w) = T(S(v)) = S(T(v)) \in \mathrm{im}(S)$.

2. Let (w, z) be an eigenvector. Then

$$\begin{pmatrix} z \\ w \end{pmatrix} = \lambda \begin{pmatrix} w \\ z \end{pmatrix}$$

so $z = \lambda^2 z$. If $z = 0$ then $w = 0$, but eigenvectors must be nonzero. Therefore $z \neq 0$ and $\lambda = \pm 1$. If $\lambda = 1$, then (w, w) is an eigenvector for $w \neq 0$. If $\lambda = -1$ then $(w, -w)$ is an eigenvector for $w \neq 0$.

3. Consider the vector $v - T(v)$. Then

$$T(v - T(v)) = T(v) - v = -(v - T(v))$$

so that $v - T(v)$ is either 0 or an eigenvector with eigenvalue -1. Since T has no such eigenvectors $v - T(v) = 0$. Therefore $T(v) = v$.

4. (a) Induction: $T(0,1) = (1,1)$ and assume that $T^n(0,1) = (F_n, F_{n+1})$ Then $T^{n+1}(0,1) = (F_{n+1}, F_{n+1} + F_n) = (F_{n+1}, F_{n+2})$.

   (b) An eigenvector must satisfy

$$T \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}$$

so

$$\begin{pmatrix} b \\ a + b \end{pmatrix} = \begin{pmatrix} \lambda a \\ \lambda b \end{pmatrix}.$$

Either $a = b = 0$ or $a \neq 0$ and $a + \lambda a = \lambda^2 a$ so $1 + \lambda = \lambda^2$. Solving for λ produces two solutions

$$\phi := \frac{1 + \sqrt{5}}{2}, \ \psi := \frac{1 - \sqrt{5}}{2}.$$

   (c) From the last part, the eigen vectors are of the form $(a, \lambda a)$ for any nonzero a. Therefore $(1, \phi)$ and $(1, \psi)$ form an eigenbasis (they're linearly independent since they correspond to different eigenvalues).

   (d) Note that $T^n(1, \phi) = \phi^n(1, \phi)$ and $T^n(1, \psi) = \psi^n(1, \psi)$. Write

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = a \begin{pmatrix} 1 \\ \phi \end{pmatrix} + b \begin{pmatrix} 1 \\ \psi \end{pmatrix}$$

for some a, b. It is not hard to see that

$$a = \frac{1}{\sqrt{5}}, \ b = -\frac{1}{\sqrt{5}}$$

. Therefore

$$\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{5}}\left(\begin{pmatrix}1\\\phi\end{pmatrix} - \begin{pmatrix}1\\\psi\end{pmatrix}\right)$$

$$\Rightarrow T^n\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{5}}\left(T^n\begin{pmatrix}1\\\phi\end{pmatrix} - T^n\begin{pmatrix}1\\\psi\end{pmatrix}\right) = \frac{1}{\sqrt{5}}\left(\phi^n\begin{pmatrix}1\\\phi\end{pmatrix} - \psi^n\begin{pmatrix}1\\\psi\end{pmatrix}\right).$$

Looking at the first coordinate and using part (a) shows that

$$F_n = \frac{1}{\sqrt{5}}(\phi^n - \psi^n).$$

(e) Since $\frac{|\psi|^n}{\sqrt{5}} < \frac{1}{2}$, the closest integer to F_n is $\frac{\phi^n}{\sqrt{5}}$.

6. Want to solve

$$\frac{d^2u}{dx^2u} = \lambda u.$$

Let

$$u = a_0 + a_1x + a_2x^2 + \cdots$$

Then

$$\frac{d^2u}{dx^2} = 2a_2 + 6a_3x + 12a_4x^2 + \cdots .$$

Comparing coefficients of the powers of x shows that

$$2a_2 = \lambda a_0$$

$$6a_3 = \lambda a_1$$

$$12a_4 = \lambda a_2$$

$$\vdots$$

$$n(n-1)a_n = \lambda a_{n-2}.$$

Therefore

$$a_n = \frac{\lambda a_{n-2}}{n(n-1)} = \frac{\lambda^2 a_{n-4}}{n(n-1)(n-2)(n-3)} = \cdots$$

$$\Rightarrow a_{2m} = \frac{\lambda^m a_0}{(2m)!},\ a_{2m+1} = \frac{\lambda^m a_1}{(2m+1)!}$$

so that

$$u = a_0 + a_1x + \frac{\lambda a_0 x^2}{2!} + \frac{\lambda a_1 x^3}{3!} + \frac{\lambda^2 a_0 x^4}{4!} + \frac{\lambda^2 a_1 x^5}{5!} + \cdots .$$

A basis for the λ eigenspace is therefore given by the following two power series

$$1 + \frac{\lambda x^2}{2!} + \frac{\lambda^2 x^4}{4!} + \cdots = \sum_{m=0}^{\infty} \frac{\lambda^m x^{2m}}{(2m)!}$$

$$x + \frac{\lambda x^3}{3!} + \frac{\lambda^2 x^5}{5!} + \cdots = \sum_{m=0}^{\infty} \frac{\lambda^m x^{2m+1}}{(2m+1)!}.$$

Therefore there's an eigenspace of dimension 2 for each $\lambda \in \mathbb{R}$. For the $\lambda = -1$ eigenspace, the Taylor series for sin and cos about $x = 0$ form a basis.