Math 110 Homework 2 SOLUTIONS

1. Let $(u_1,\dots,u_n)$ be a basis for U and $(w_1,\dots,w_m)$ a basis for W. If $U+W$ is a direct sum, then any vector in $U+W$ can be written uniquely as $u+w$ where $u\in U$ and $w\in W$. It therefore can be written uniquely as $a_1u_1+\cdots a_nu_n+b_1w_1+\cdots+b_mw_m$. Therefore $(u_1,\dots,u_n,w_1,\dots,w_m)$ is a basis of $U+W$ if $U+W$ is a direct sum.

   If $U\cap W=\{0\}$ then $U+W$ is a direct sum. If U and W are both dimension 5, then by the last paragraph $U\cap W=\{0\}$ implies that $U+W$ has dimension 10.

   You cannot have a 10-dimensional subspace of $\mathbb{R}^9$, since a basis for any subspace can be extended to a basis of the whole space.

2. Let $V=\mathbb{R}^2$ and consider the three 1-dimensional subspaces $U_1:=\mathrm{Span}(e_1)$, $U_2:=\mathrm{Span}(e_2)$, and $U_3:=\mathrm{Span}(e_1+e_2)$. Then $U_1+U_2+U_3=\mathbb{R}^2$ so
$$\dim(U_1+U_2+U_3)=2.$$
   Each intersection $U_i\cap U_j$ is the zero subspace, so
$$\dim(U_1)+\dim(U_2)+\dim(U_3)-\dim(U_1\cap U_2)-\dim(U_2\cap U_3)-\dim(U_3\cap U_1)+\dim(U_1\cap U_2\cap U_3)$$
$$=1+1+1-0-0-0-0=3.$$
   This provides a counterexample.

3. Let $(u_1,\dots,u_k)$ be a basis of U. Extend this to a basis of V: $(u_1,\dots,u_k,v_{k+1},\dots,v_n)$. A linear map is determined by where it sends basis vectors. Define $T(u_i)=S(u_i)$ and $T(v_j)=0$. Then for $u=a_1u_1+\cdots+a_ku_k$,
$$T(u)=a_1T(u_1)+\cdots+a_kT(u_k)=a_1S(u_1)+\cdots+a_kS(u_k)=S(u).$$

4. Suppose T is a scalar multiple of the identity matrix. Then $TS(v)=c\,\mathrm{id}_V\,S(v)=cS(v)=S(cv)=S(c\,\mathrm{id}_V(v))=ST(v)$. This proves one direction.

   To prove the other direction, suppose that T is such that $TS=ST$ for all $S\in\mathcal{L}(V,V)$. After fixing a basis $(v_1,\dots,v_n)$ of V let M be the matrix representing T. It is enough to show that if $MN=NM$ for all $n\times n$ matrices N then M is a scalar multiple of the identity matrix.

   Let $E(i,j)$ be the $n\times n$ matrix which is filled with zeroes except for a single 1 in the ijth entry. Note that $E(i,j)M$ is the matrix whose ith row is the jth row of M, and all other rows are zero:
$$(E(i,j)M)_{\ell k}=\sum_{p=1}^{n}E(i,j)_{\ell p}M_{pk}=\begin{cases}M_{jk} & \ell=i\\ 0 & \text{otherwise}\end{cases}.$$

Similarly, $ME(i,j)$ is the matrix whose jth column is the ith column of M, and all other columns are 0.

Suppose that M is such that $MN = NM$ for all $n \times n$ matrices N. Then in particular $ME(i,j) = E(i,j)M$ for all i,j. The equation $ME(i,i) = E(i,i)M$ shows that the ith row of M must 0 except for the ith position and the ith column must be 0 except for the ith position. Therefore M must be diagonal. Let $(\lambda_1, \ldots, \lambda_n)$ be the diagonal entries of M. Then $ME(i,j) = \lambda_i E(i,j)$ and $E(i,j)M = \lambda_j E(i,j)$, so all the diagonal entries must be the same.

Therefore M is a scalar multiple of the identity matrix. The linear transformation it represents is a scalar multiple of the identity.

5. In class I showed that $\dim \mathcal{L}(V,W) = (\dim V)(\dim W)$. Therefore $\dim(\mathcal{L}(\mathbb{F},V)) = (\dim \mathbb{F})(\dim V) = \dim V$. Since $\mathcal{L}(\mathbb{F},V)$ and V have the same dimension, they are isomorphic.

6. Let
$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}$$
$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$
$$c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}.$$

Think of the matrix A a linear map from $\mathbb{F}^n$ to itself. The equation in (a) can be restated as $x \in \ker(A) \Rightarrow x = 0$, which is equivalent to A being injective. The equation in (b) can be restated as: given $c \in \mathbb{F}^n$ there exists $x \in \mathbb{F}^n$ such that $Ax = c$, which is the statement that A is surjective.

If $A : \mathbb{F}^n \to \mathbb{F}^n$, surjective implies injective and injective implies surjective. If A is surjective, then $\dim(\mathrm{im}(A)) = n$ so rank-nullity implies that $\dim(\ker(A)) = 0$ which implies that A is injective. If A is injective, then $\dim(\ker(A)) = 0$ so rank-nullity implies that $\dim(\mathrm{im}(A)) = n$ which implies that A is surjective.

Therefore since (a) is the statement that A is injective and (b) is the statement that A is surjective, one implies the other.