Math 110 Homework 1 SOLUTIONS

1. Let W_1 and W_2 be two subspaces of V and suppose that one is not contained in the other. To show that $W_1 \cup W_2$ is not a subspace, it is enough to find vectors $w_1, w_2 \in W_1 \cup W_2$ such that $w_1 + w_2 \notin W_1 \cup W_2$.

   Since W_1 is not contained in W_2 there exists $w_1 \in W_1$ which is not in W_2. Similarly there exists $w_2 \in W_2$ which is not in W_1. Suppose that $w_1 + w_2$ were in $W_1 \cup W_2$. Then $w_1 + w_2$ would be in either W_1 or W_2. If it were in W_1, then $(w_1 + w_2) - w_1 = w_2$ would be in W_1, contradicing the assumption on w_2. Similarly, if $w_1 + w_2$ were in W_1, then $(w_1 + w_2) - w_1 = w_1$ would be in W_2, contradicting the assumption on w_1. Thus $w_1 + w_2 \notin W_1 \cup W_2$.

   The other direction of the "if and only if" is much easier. If $W_1 \subset W_2$, then $W_1 \cup W_2 = W_2$ and is thus a subspace. Similarly for $W_2 \subset W_1$.

2. To show that $\{v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4\}$ spans V, you must show that given $v \in V$, you can write

   $$v = a_1(v_1 - v_2) + a_2(v_2 - v_3) + a_3(v_3 - v_4) + a_4 v_4$$

   for some $a_1, a_2, a_3, a_4 \in \mathbb{F}$. Since $\{v_1, v_2, v_3, v_4\}$ spans, then

   $$v = b_1 v_1 + b_2 v_2 + b_3 v_3 + b_4 v_4$$

   for some $b_1, b_2, b_3, b_4 \in \mathbb{F}$. Define

   $$a_1 := b_1,\ a_2 := b_1 + b_2,\ a_3 := b_1 + b_2 + b_3,\ a_4 := b_1 + b_2 + b_3 + b_4.$$

   With these values

   $$a_1(v_1 - v_2) + a_2(v_2 - v_3) + a_3(v_3 - v_4) + a_4 v_4$$
   $$= b_1(v_1 - v_2) + (b_1 + b_2)(v_2 - v_3) + (b_1 + b_2 + b_3)(v_3 - v_4) + (b_1 + b_2 + b_3 + b_4)v_4.$$
   $$= b_1 v_1 + b_2 v_2 + b_3 v_3 + b_4 v_4 = v.$$

3. To show that $\{v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4\}$ is linearly independent, you have to show that

   $$a_1(v_1 - v_2) + a_2(v_2 - v_3) + a_3(v_3 - v_4) + a_4 v_4 = 0 \Rightarrow (a_1, a_2, a_3, a_4) = (0, 0, 0, 0).$$

   Suppose that

   $$a_1(v_1 - v_2) + a_2(v_2 - v_3) + a_3(v_3 - v_4) + a_4 v_4 = 0.$$

   Then

   $$a_1 v_1 + (a_2 - a_1)v_2 + (a_3 - a_2)v_3 + (a_4 - a_3)v_4 = 0.$$

   The linear independence of $\{v_1, v_2, v_3, v_4\}$ implies that

   $$a_1 = a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = 0$$

   and this implies that

   $$a_1 = a_2 = a_3 = a_4 = 0.$$

4. Similar proofs to the previous two propositions show that $\{v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4\}$ is linearly independent and spanning. Hence it is a basis.

5. The following axioms hold by properties of the complex numbers:

$$\alpha + \beta = \beta + \alpha,\ \ \alpha, \beta \in \mathbb{C}$$

$$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma,\ \ \alpha, \beta, \gamma \in \mathbb{C}$$

$$(ab)\alpha = a(b\alpha),\ \ \alpha \in \mathbb{C},\ \ a, b \in \mathbb{R}$$

$$(a + b)(\alpha + \beta) = a\alpha + a\beta + b\alpha + b\beta,\ \ \alpha, \beta \in \mathbb{C},\ \ a, b \in \mathbb{R}.$$

$$1\alpha = \alpha.$$

   0 serves as the additive identity in $\mathbb{C}$ and satisfies $0+\alpha = \alpha$ for all $\alpha \in \mathbb{C}$. $-\alpha$ serves as the additive inverse in $\mathbb{C}$ and satisfies $\alpha + (-\alpha) = 0$ for all α.

6. Two functions f, g are linearly independent over $\mathbb{R}$ if

$$af + bg = 0 \Rightarrow (a, b) = (0, 0)$$

   where $a, b \in \mathbb{R}$ and the equality $af + bg = 0$ is an equality of functions. In particular, it holds for all x: $af(x) + bg(x) =$ for all $x \in \mathbb{R}$.

   Suppose that

$$a \sin(x) + b \cos(x) = 0$$

   for all $x \in \mathbb{R}$. Set $x = 0$ to see that $b = 0$ and set $x = \frac{\pi}{2}$ to see that $a = 0$.