**Math 110 Final (SOLUTIONS)**
August 9, 2018

1. The span of the columns is the image. Therefore the image is 1-dimensional. By rank-nullity, $\dim(\ker(T)) = n - 1$.

2. Suppose $v \in \ker(T) \cap \operatorname{im}(T)$. Then $T(v) = 0$ and $v = T(w)$ for some $w \in V$. Then $0 = T(v) = T^2(w) = T(w) = v$.

3. (a) The axioms of an inner product imply that

$$\langle v + w, v_0\rangle = \langle v, v_0\rangle + \langle w, v_0\rangle$$

so that

$$T(v + w) = \langle v + w, v_0\rangle v_0 = \langle v, v_0\rangle v_0 + \langle w, v_0\rangle v_0 = T(v) + T(w).$$

and the axioms of an inner product also imply that

$$\langle cv, v_0\rangle = c\langle v, v_0\rangle$$

so that

$$T(cv) = \langle cv, v_0\rangle v_0 = c\langle v, v_0\rangle v_0 = cT(v).$$

(b) If $v_0 \neq 0$, then $v_0$ is an eigenvector of $T$ (with eigenvalue $\|v_0\|^2$). If $v_0 = 0$, then $T$ is the zero map, so any nonzero vector is an eigenvector (with eigenvalue 0).

4. Given a linear relation

$$a_0 v + a_1 T(v) + \cdots + a_{k-1}T^{k-1}(v) = 0$$

apply $T^{k-1}$ to each side to get that

$$a_0 T^{k-1}(v) + a_1 T^k(v) + \cdots + a_{k-1}T^{2k-1}(v) = 0.$$

Then all but the first term are 0, so

$$a_0 T^{k-1}(v) = 0$$

so either $a_0 = 0$ or $T^{k-1}(v) = 0$. The latter is nonzero by construction. Therefore $a_0 = 0$. Therefore

$$a_1 T(v) + a_2 T^2(v) + \cdots + a_{k-1}T^{k-1}(v) = 0.$$

Apply $T^{k-2}$ to each side to similarly get that $a_1 = 0$. Repeat in this manner until you've shown that each $a_i$ is zero.

5. The singular vectors in the domain are the eigenvectors of $T^*T$.

$$T^*T = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}.$$

The eigenvalues are the roots of $(5-\lambda)^2 - 1 = 24 - 10\lambda + \lambda^2 = (4-\lambda)(6-\lambda)$. The eigenvectors are therefore

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The singular values are $s_1 = \sqrt{6}$ and $s_2 = \sqrt{4} = 2$. The singular vectors are the normalizations of $T(v_1)$ and $T(v_2)$, so

$$w_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$w_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

To complete this to an orthonormal basis of $\mathbb{R}^3$, add

$$w_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

6. Since $T$ is normal let $(v_1, \ldots, v_n)$ be an orthonormal eigenbasis with eigenvalues $\lambda_1, \ldots, \lambda_n$. Write

$$v = a_1 v_1 + \cdots + a_n v_n$$

so that

$$T^k(v) = a_1 T^k(v_1) + \cdots + a_n T^k(v_n) = \sum_{i=1}^n \lambda_i^k a_i v_i.$$

Therefore

$$\|T_k(v)\|^2 = \sum_{i=1}^n |\lambda_i|^{2k} |a_i|^2.$$

As $k \to \infty$, $\lambda_i^k \to 0$ if $i \geq 2$. Recall that $\lambda_1 = 1$. Therefore

$$\lim_{k \to \infty} \|T^k(v)\|^2 = |a_1|^2.$$

This is equal 0 only if $a_1 = 0$, so the vectors for which the limit is zero is the span of $\{v_2, v_3, \ldots, v_n\}$.

7. If $T$ is diagonalizable with $(v_1, \ldots, v_n)$ an eigenbasis then $STS^{-1}$ is diagonalizable with $(Sv_1, \ldots, Sv_n)$ an eigenbasis. Therefore

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$$

is diagonalizable with eigenbasis

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The two eigenspaces are not orthogonal. Therefore while

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

is self-adjoint,

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

is not, since there cannot be two orthogonal eigenvectors corresponding to different eigenvalues.

8. It is actually easier to do the projection to $U^\perp$ first. A basis for $U^\perp$ consists of

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

and a vector

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

such that

$$a + 2b = 0, \; 2b + c = 0.$$

Set $d = 0$ and note that $a = c = -2b$. Therefore

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{3} \begin{pmatrix} 2 \\ -1 \\ 2 \\ 0 \end{pmatrix}$$

forms a basis of $U^\perp$. The projection of $(3, 6, 9, 0)$ to $U^\perp$ is its projection to the span of $(2, -1, 2, 0)$, since it's orthogonal to $(0, 0, 0, 1)$. Therefore the projection is

$$\frac{1}{9} \left( \begin{pmatrix} 3 \\ 6 \\ 9 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 2 \\ 0 \end{pmatrix} \right) \begin{pmatrix} 2 \\ -1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 4 \\ 0 \end{pmatrix}.$$

And so the projection of $(3, 6, 9, 0)$ to $U$ is

$$\begin{pmatrix} 3 \\ 6 \\ 9 \\ 0 \end{pmatrix} - \begin{pmatrix} 4 \\ -2 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 8 \\ 5 \\ 0 \end{pmatrix}.$$