**Math 110 Final (PRACTICE)**

110 Minutes

1. Let $v_1, \ldots, v_n$ be a set of vectors in an inner product space. Define what it means for $v_1, \ldots, v_n$ to be linearly independent. Define what it means for $v_1, \ldots, v_n$ to be orthonormal.

2. In $\mathbb{R}^3$ , let
$$U = \operatorname{Span}\begin{pmatrix}1\\1\\1\end{pmatrix}.$$
Find an orthonormal basis for $U^\perp$.

3. Let $M$ be an $n \times n$ matrix with entries in $\mathbb{F}$. Suppose that its characteristic polynomial is $\det(tI - M) = (t-1)^n$.
   (a) Prove or disprove: the rows of $M$ are linearly independent.
   (b) Prove or disprove: $M$ is the identity matrix.

4. Let $V$ be a finite-dimensional complex inner product space.
   (a) Prove or disprove: Self-adjoint operators on $V$ form a subspace of $\mathcal{L}(V, V)$.
   (b) Prove or disprove: Isometries of $V$ form a subspace of $\mathcal{L}(V, V)$.

5. Let $V$ be a complex inner product space. Prove that
$$\langle v, w\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \|v + i^k w\|^2.$$
Here $i = \sqrt{-1}$.

6. Let $T : V \to V$ be a linear map of complex finite-dimensional vector spaces. Suppose $T$ is not invertible. Does there exist a $\lambda \in \mathbb{C}$ such that $T - \lambda \operatorname{id}_V$ is invertible? Explain.

7. Consider $\mathbb{R}^2$ and $\mathbb{R}^3$ with the dot product. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given, with respect to the standard basis, by the matrix
$$\begin{pmatrix}1 & 0\\1 & 0\\0 & 1\end{pmatrix}.$$
What are the singular vectors and singular values of $T$?

8. Let $V$ be a finite-dimensional inner product space and suppose that $S$ and $T$ are self-adjoint. Prove that if $ST = TS$ then there exists an orthonormal basis $(v_1, \ldots, v_n)$ of $V$ which is an eigenbasis for both $S$ and $T$. (hint: the $\lambda$-eigenspace for $S$ is invariant for $T$ and vice versa)