Math 110 Final (PRACTICE: 3/4 length) SOLUTIONS

1. The image is the span of the columns and hence 1-dimensional. Therefore the kernel is 2-dimensional. $(1,1,1)$ is an eigenvector with eigenvalue 3 and the kernel is spanned by $(1,-1,0)$ and $(0,1,-1)$. Therefore the eigenvectors of eigenvalue 3 are the nonzero vectors of the form (a,a,a) and the eigenvectors with eigenvalue 0 are the nonzero vectors of the form (a,b,c) with $a+b+c=0$.

2. Let v be a basis of $\operatorname{im}(T)$. In particular, $v \neq 0$. Then $T(v) = av$, since $T(v) \in \operatorname{im}(T)$. Therefore v is an eigenvector.

3. Suppose a linear relation

$$a_1 v_i + a_2 v_j = 0.$$

Take an inner product on both sides with v_i and and inner product on both sides with v_j to get two equations

$$a_1\langle v_i, v_i\rangle + a_2\langle v_j, v_i\rangle = 0$$

$$a_1\langle v_i, v_j\rangle + a_2\langle v_j, v_j\rangle = 0$$

i.e.,

$$2a_1 - a_2 = 0$$

$$-a_1 + 2a_2 = 0.$$

Solving these two equations shows that $a_1 = a_2 = 0$ therefore v_i and v_j are linearly independent.

The three vectors are linearly dependent because $v_i + v_j + v_k = 0$. To see this, note that

$$\langle v_i+v_j+v_k, v_i+v_j+v_k\rangle = \|v_i\|^2+\|v_j\|^2+\|v_k\|^2+2\langle v_i, v_k\rangle+2\langle v_i, v_j\rangle+2\langle v_j, v_k\rangle = 0.$$

The norm of a vector is 0 only if that vector is 0.

4. Perform Gram-Schmidt on $1, x^2$. Normalize 1 to get the function

$$u_1 = \frac{1}{\sqrt{2}}.$$

Then $x^2 - \langle x^2, u_1\rangle u_1 = x^2 - \frac{1}{2}\int_{-1}^{1} x^2 dx = x^2 - \frac{1}{3}$. The norm squared of this is

$$\left\langle x^2 - \frac{1}{3}, x^2 - \frac{1}{3}\right\rangle = \int_{-1}^{1}\left(x^2 - \frac{1}{3}\right)^2 dx = \frac{45}{8}.$$

Therefore

$$u_2 = \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right).$$

The projection of x^4 onto the space spanned by u_1 and u_2 is

$$\langle x^4, u_1\rangle u_1 + \langle x^4, u_2\rangle u_2 = \frac{1}{2}\int_{-1}^{1} x^4 dx + \frac{45}{8}\int_{-1}^{1} x^4\left(x^2 - \frac{1}{3}\right)dx$$

$$= \frac{1}{5} + \frac{6}{7}\left(x^2 - \frac{1}{3}\right)$$

5. Let U_1, U_2, and U_3 be subspaces each of dimension 4. Since $\dim(U_1 + U_2) = \dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2)$ then

$$5 \geq \dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2)$$

so

$$\dim(U_1 \cap U_2) \geq 3.$$

Similarly,

$$5 \geq \dim(U_1 \cap U_2) + \dim(U_3) - \dim(U_1 \cap U_1 \cap U_3) = \dim(U_1 \cap U_2) + 4 - \dim(U_1 \cap U_2 \cap U_3)$$

$$\geq 3 + 4 - \dim(U_1 \cap U_2 \cap U_3).$$

Therefore

$$\dim(U_1 \cap U_2 \cap U_3) \geq 2.$$

In particular, $U_1 \cap U_2 \cap U_3$ contains a line.

6. Let $\phi \in \mathcal{M}_{n,k}$. Then ϕ applied to k arbitrary vectors is

$$\phi\left(\sum_{i_1=1}^{n} M_{i_1 1} e_{i_1}, \ldots, \sum_{i_k=1}^{n} M_{i_k k} e_{i_k}\right)$$

which by multilinearity is

$$\sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} M_{i_1 1} \cdots M_{i_k k}\phi(e_{i_1}, \ldots, e_{i_k}).$$

Note that there are n^k different sequences $(i_1, i_2, \ldots, i_k)$. ϕ is determined by its values on the n^k k-tuples of standard basis vectors $(e_{i_1}, \ldots, e_{i_k})$. Defining ϕ to 1 on one of these and 0 on the other $n^k - 1$ defines n^k linearly independent vectors in $\mathcal{M}_{n,k}$. Explicitly:

$$\phi_{i_1,i_2,\ldots,i_k}(e_{j_1}, \ldots, e_{j_k}) := \begin{cases} 1 & i_1 = j_1, \ldots, i_k = j_k \\ 0 & \text{otherwise} \end{cases}.$$

Since any ϕ can be written as a linear combination of the $\phi_{i_1,\ldots,i_k}$, these form a basis. Therefore $\dim(\mathcal{M}_{n,k}) = n^k$.