**Math 110 Final**
August 9, 2018
110 Minutes

Name:________________________

| | |
|---|---|
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| 6 | |
| 7 | |
| 8 | |

1. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be given, with respect to the standard basis, by an $m \times n$ matrix filled with 1s:

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}.$$

What is $\dim(\ker(T))$?

2. Let $T : V \to V$ be a linear map such that $T^2 = T$. Prove that $\ker(T) \cap \operatorname{im}(T) = \{0\}$.

3. Let $V$ be an inner product space. Let $v_0$ be a vector in $V$. Consider the following map

$$T(v) = \langle v, v_0 \rangle v_0.$$

(a) Prove that $T$ is a linear map.

(b) Prove that $T$ has an eigenvector.

4. Let $V$ be a vector space and $T : V \to V$ a linear map. Suppose that $v \in V$ is such that $T^{k-1}(v) \neq 0$ but $T^k(v) = 0$. Prove that

$$\{v, T(v), T^2(v), T^3(v), \dots, T^{k-1}(v)\}$$

is a linearly independent set.

5. Consider $\mathbb{R}^2$ and $\mathbb{R}^3$ with the dot product. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given, with respect to the standard basis, by the following matrix:

$$\begin{pmatrix} 2 & 0 \\ 1 & 1 \\ 0 & 2 \end{pmatrix}.$$

Find the singular values and singular vectors of $T$.

6. Let $T : V \to V$ be a normal operator on a complex inner product space $V$ with $\dim(V) = n$. Call its eigenvalues $\lambda_1, \ldots, \lambda_n$. Suppose that $|\lambda_1| = 1$ and $|\lambda_i| < 1$ for $i \geq 2$. Which vectors $v \in V$ satisfy

$$\lim_{k\to\infty} \|T^k v\| = 0?$$

Justify your answer.

7. Let $V$ be a finite-dimensional real inner product space. Prove or disprove: if $S : V \to V$ is invertible and $T : V \to V$ is self-adjoint, then $STS^{-1}$ is self-adjoint.

8. Consider $\mathbb{R}^4$ with the dot product. Let $U \subset \mathbb{R}^4$ be the span of

$$\begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \end{pmatrix}.$$

(a) What is the projection of $\begin{pmatrix} 3 \\ 6 \\ 9 \\ 0 \end{pmatrix}$ to $U$?

(b) What is the projection of $\begin{pmatrix} 3 \\ 6 \\ 9 \\ 0 \end{pmatrix}$ to $U^\perp$?