Math 110

Scraps From Drafting the Final

1. Give an example of a vector space V and a linear map $T : V \to V$ whose eigenvalues include all nonnegative integers.

2. Define $V^\vee = \mathcal{L}(V, \mathbb{F})$. Let V and W be finite-dimensional vector spcaes. If $T : V \to W$ is a linear map, define $T^\vee : W^\vee \to V^\vee$ by $T^\vee(\phi) = \phi \circ T$.

   (a) Show that if T is surjective then $T^\vee$ is injective.

   (b) Show that if T is injective then $T^\vee$ is surjective.

3. Let V be a finite-dimensional inner product space and let $T : V \to V$ be a self-adjoint isometry. What are the possible eigenvalues of T?

4. Let V be a real inner product space. Prove that self-adjoint operators $T : V \to V$ form a subspace of $\mathcal{L}(V, V)$. What is the dimension of this subspace?

5. Let V be a complex inner product space. Suppose T is self-adjoint and that $T^N = \mathrm{id}_V$ for some $N \geq 0$. What are the possible eigenvalues for T?

6. Consider $\mathbb{C}$ as a vector space over $\mathbb{R}$. Define an inner product on $\mathbb{C}$ as

$$\langle a + bi, c + di \rangle = ac + bd.$$

   For $z \in \mathbb{C}$, let $T_z : \mathbb{C} \to \mathbb{C}$ be multiplication by z. For what values of z is T_z self-adjoint? For what values of z is T_z an isometry?

7. Consider $\mathbb{R}^2$ with the dot product. Find a diagonalizable linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that does not have an orthogonal eigenbasis.

8. (a) Give an example of a linear transformation from a complex vector space to itself for which every vector is an eigenvector.

   (b) Give an example of a linear transformation from a complex vector space to itself that is not diagonalizable.