Math 110

August 2, 2018

The Determinant (SOLUTIONS)

1. e, (12), (13), (14), (23), (24), (34), (12)(34), (13)(24), (14)(23), (123), (132), (124), (142), (134), (143), (234), (243), (1234), (1243), (1324), (1342), (1423), (1432)

2. Suppose the permutation is (ab) with $a < b$. In the denominator, there are two kinds of factors: those that don't involve a or b and those that do. The former simply cancel with factors in the numerator. The latter split into two kinds: the one term that involves both a and b and the many terms that involve just one of a or b. The first kind cancels with a term in the numerator to get -1:

$$\frac{(x_b - x_a)}{(x_a - x_b)} = -1.$$

The many terms that involve just one of a or b come in pairs

$$(x_a - x_i),\ (x_b - x_i),\ \text{if } a < b < i$$

$$(x_a - x_i),\ (x_i - x_b),\ \text{if } a < i < b$$

$$(x_i - x_a),\ (x_i - x_b),\ \text{if } i < a < b.$$

In all these cases, you can check that these pairs cancel with pairs in the numerator either both as $+1$ or both as -1. Therefore in all the cancelation between numerator and denominator, there is an odd number negative signs.

3.

$$e \mapsto \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix},\ (12) \mapsto \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix},\ (13) \mapsto \begin{pmatrix} & & 1 \\ & 1 & 1 \\ 1 & & \end{pmatrix}$$

$$(23) \mapsto \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix},\ (123) \mapsto \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix},\ (132) \mapsto \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}.$$

Because

$$\det(e_1, \dots, e_n) = \operatorname{sign}(\sigma) \det(e_{\sigma(1)}, \dots, e_{\sigma(n)})$$

the determinant of such a matrix is $\operatorname{sign}(\sigma)^{-1} = \operatorname{sign}(\sigma)$.

4. An alternating map is zero if two of its arguments are equal. Using multilinearity, you can write $\phi(v_1, v_2, v_3)$ as a linear combination of $\phi(e_i, e_j, e_k)$ where e_i, e_j and e_k are standard basis vectors. Since there are only two standard basis vectors, there must be some repeat and $\phi = 0$.

5.

$$2\det\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix} - 0\det\begin{pmatrix}0 & 1\\ 0 & 2\end{pmatrix} + 1\det\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = 2(2)+0(0)+1(-1) = 3.$$

6. This matrix is obtained from the previous by switching two columns, so its determinant is -3.

7.

$$1\det\begin{pmatrix}5 & 6\\ 8 & 9\end{pmatrix} - 2\det\begin{pmatrix}4 & 6\\ 7 & 9\end{pmatrix} + 3\det\begin{pmatrix}4 & 5\\ 7 & 8\end{pmatrix} = 1(-3)-2(-6)+3(-3) = 0$$

8.

9. No, if I is a 2×2 identity matrix then $\det(I+(-I)) = 0$ but $\det(I) = \det(-I) = 1$.

10.

$$\det\begin{pmatrix}1-\lambda & 0 & -1\\ 1 & 2-\lambda & 1\\ 0 & 0 & 2-\lambda\end{pmatrix} = (2-\lambda)^2(1-\lambda)$$

(for an easy way to compute this, you can use 12 below). Then solving

$$a - c = \lambda a$$

$$a + 2b + c = \lambda b$$

$$2c = \lambda c$$

for $\lambda = 1$ gives a solution of the form

$$\begin{pmatrix}a\\ -a\\ 0\end{pmatrix}.$$

Solving for $\lambda = 2$ gives

$$\begin{pmatrix}a\\ b\\ -a\end{pmatrix}.$$

Varying a and b gives a 2-dimensional subspace of $\mathbb{R}^3$. Thus the eigenvectors are of the form

$$\begin{pmatrix}a\\ -a\\ 0\end{pmatrix}$$

for some $a \in \mathbb{R}$ or

$$\begin{pmatrix}a\\ b\\ -a\end{pmatrix}$$

for some $a, b \in \mathbb{R}$.

For the next matrix, a computation shows that

$$\det \begin{pmatrix} 2-\lambda & 4 & 4 \\ 4 & 2-\lambda & -4 \\ 6 & 6 & -\lambda \end{pmatrix} = -\lambda(\lambda-6)(\lambda+2).$$

Therefore the eigenvalues are $6, -2, 0$. One solves

$$2a + 4b + 4c = \lambda a$$
$$4a + 2b - 4c = \lambda b$$
$$6a + 6b = \lambda c$$

for each of $\lambda = 6, -2, 0$ to get three eigenspaces

$$\text{Span} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \ \text{Span} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \ \text{Span} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}.$$

11. No. Consider the case where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ D = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $\det(M) = -1$ but $\det(A)\det(D) - \det(B)\det(C) = 0$.

12. In the sum of permutations definition of the determinant, each term can be thought of as a path through the columns of the matrix, from left to right, that touches each row exactly once. The only paths that don't touch the zero area of M are the ones that go through A and then go through D. If A is $n \times n$ and D is $m \times m$, then each path corresponds to a permutation of the form $\sigma\tau$ where σ permutes $\{1, \ldots, n\}$ and τ permutes $\{n+1, \ldots, n+m\}$. Note that $\text{sign}(\sigma\tau) = \text{sign}(\sigma)\,\text{sign}(\tau)$. Write $S_{n,m}$ to denote the permutations that permute the first n and the last m separately. If $\sigma' \in S_{n,m}$ then $\sigma' = \sigma\tau$ as above.

Then

$$\begin{aligned} \det(M) &= \sum_{\sigma' \in S_{n,m}} \text{sign}(\sigma') M_{\sigma'(1)1} \cdots M_{\sigma'(n+m),n+m} \\ &= \sum_{\sigma,\tau} \text{sign}(\sigma\tau) M_{\sigma\tau(1)1} \cdots M_{\sigma\tau(n+m),n+m} \\ &= \sum_{\sigma,\tau} \text{sign}(\sigma)\,\text{sign}(\tau) M_{\sigma(1)1} \cdots M_{\sigma(n)n} M_{\tau(n+1),n+1} \cdots M_{\tau(n+m),n+m} \\ &= \left(\sum_{\sigma} \text{sign}(\sigma) M_{\sigma(1)1} \cdots M_{\sigma(n)n}\right) \left(\sum_{\tau} \text{sign}(\tau) M_{\tau(n+1),n+1} \cdots M_{\tau(n+m),n+m}\right) \\ &= \det(A)\det(D). \end{aligned}$$