Math 110
August 2, 2018
The Determinant

1. In cycle notation, write out all 24 permutations of $\{1, 2, 3, 4\}$.

2. Using the definition
$$\operatorname{sign}(\sigma) = \frac{\prod_{i<j}(x_{\sigma(i)} - x_{\sigma(j)})}{\prod_{i<j}(x_i - x_j)}$$
prove that if σ is a transposition then $\operatorname{sign}(\sigma) = -1$. Remember that a transposition is a permutation that switches exactly two elements of $\{1, \ldots, n\}$.

3. For each $\sigma \in S_3$, write down the matrix for the linear map that takes e_i to $e_{\sigma(i)}$. What is the determinant of such a matrix?

4. Prove that the vector space of alternating multilinear maps $\mathbb{F}^2 \times \mathbb{F}^2 \times \mathbb{F}^2 \to \mathbb{F}$ is the zero vector space.

5. Compute the determinant of
$$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

6. Compute the determinant of
$$\begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

7. Compute the determinant of
$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

8. Compute the characteristic polynomials of the above three matrices.

9. Is it true that $\det(M + N) = \det(M) + \det(N)$?

10. Find the eigenvalues and eigenvectors of
$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 4 & 4 \\ 4 & 2 & -4 \\ 6 & 6 & 0 \end{pmatrix}.$$

11. Consider a 4×4 matrix of the form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A, B, C, and D are 2×2 matrices. Is $\det(M) = \det(A)\det(D) - \det(B)\det(C)$.?

12. Consider an arbitrary square matrix

$$M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$$

where A and D are square blocks. Prove that $\det(M) = \det(A)\det(D)$.

13. The trace of a matrix M, $\mathrm{tr}(M)$, is the sum of the diagonal entries.

    (a) Prove that $\mathrm{tr}(ABC) = \mathrm{tr}(CAB) = \mathrm{tr}(BCA)$.

    (b) Prove that, like the determinant, tr is invariant under conjugation: $\mathrm{tr}(PMP^{-1}) = \mathrm{tr}(M)$.

    (c) Let $M(t)$ be a differentiable 1-parameter family of matrices such that $M(0) = I$. Prove that

    $$\frac{d}{dt}\det(M(t))|_{t=0} = \mathrm{tr}\left(\frac{dM}{dt}|_{t=0}\right).$$

    Moral: trace is the infinitesimal version of determinant.