Math 110

July 31, 2018

Singular Value Decomposition 2 (SOLUTIONS)

1.

$$T^*T = \begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix}.$$

The eigenvalues are solutions to $(5-\lambda)^2 - 9 = 0$ so $\lambda = 8, 2$. To find eigenvectors of T^*T, solve

$$\begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 8a \\ 8b \end{pmatrix}$$

and

$$\begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a \\ 2b \end{pmatrix}$$

to get

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \ \lambda = 8 \text{ and } \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ \lambda = 2.$$

Normalize these to get v_1 and v_2:

$$v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \ v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Both of these have nonzero eigenvalues, so set

$$w_1 = \frac{T(v_1)}{\sqrt{8}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$w_2 = \frac{T(v_2)}{\sqrt{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The singular values are $\sqrt{8}$ and $\sqrt{2}$.

2.

$$T^*T = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

is an orthonormal eigenbasis with the vectors corresponding to the nonzero eigenvalues first. Then

$$T(v_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Set

$$w_1 = \frac{T(v_1)}{\sqrt{1}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Since $T(v_2) = 0$, you don't construct w_2 in this way. Simply extend w_1 to an orthonormal basis (w_1, w_2), i.e. let

$$w_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The singular values are 1 and 0.

3.

$$T^*T = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix}.$$

The eigenvalues are solutions to $(\lambda - 5)^2 - 25$, so they are 10 and 0. An eigenvector corresponding to 10 is

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and an eigenvector for 0 is

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Normalize these to get v_1 and v_2:

$$v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Set

$$w_1 = \frac{T(v_1)}{\sqrt{10}} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Since v_2 corresponds to a zero eigenvalue, define w_2 so that (w_1, w_2) is an orthonormal basis. So let

$$w_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

The singular values are $\sqrt{10}$ and 0.

4.

$$T^*T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

The eigenvalues are 3, 1, and 0. By solving

$$a + b = \lambda a$$

$$a + 2b + c = \lambda b$$

$$b + c = \lambda c$$

for $\lambda = 3, 1, 0$ you get three eigenvectors which, when normalized, are

$$v_1 = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\2\\1\end{pmatrix},\ v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix},\ v_3 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\1\end{pmatrix}.$$

Set

$$w_1 = \frac{T(v_1)}{\sqrt{3}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$w_2 = \frac{T(v_2)}{\sqrt{1}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}.$$

Since (w_1, w_2) forms a basis of $\mathbb{R}^2$, no more w_is are necessary. The singular values are $\sqrt{3}$ and 1.

5. Since $(w_1, \ldots, w_n)$ is an orthonormal basis of W and $(v_1, \ldots, v_n)$ is an orthonormal basis of V, you can write

$$T^* w_i = \sum_j a_j v_j.$$

Take an inner product with v_k on each side to see that

$$a_k = \langle T^* w_i, v_k\rangle = \langle w_i, T v_k\rangle.$$

But since $T(v_k) = s_k w_k$, then

$$\langle w_i, T v_k\rangle = \langle w_i, s_k w_k\rangle = \overline{s_k}\langle w_i, w_k\rangle$$

which (note that the singular values are real) is s_i if $i = k$ and 0 otherwise. Hence

$$T^* w_i = s_i v_i.$$

Hence the singular vectors for T^* are $(w_1, \ldots, w_m)$ and $(v_1, \ldots, v_n)$ and the singular values are $s_1, \ldots, s_{\min(m,n)}$. These are the same as for T, except the roles of v_i and w_i are switched.