Math 110

July 30, 2018

Singular Value Decomposition

This handout walks through a proof of the singular value decomposition:

Theorem 1 (Singular Value Decomposition)**.** *Let V and W be finite-dimensional inner product spaces and let $T : V \to W$ be any linear map. Then there exist orthonormal bases $(v_1, \dots, v_n)$ of V and $(w_1, \dots, w_m)$ of W and nonnegative numbers s_i such that*

$$T(v) = \sum_i s_i \langle v, v_i \rangle w_i.$$

Here the sum is over $1 \le i \le n$ if $m \ge n$ and is over $1 \le i \le m$ if $m \le n$.

1. Prove that T^*T is self-adjoint. Why does it have an orthonormal eigenbasis $(v_1, \dots, v_n)$?

2. Prove that T^*T is positive semidefinite. What does this say about its eigenvalues $\lambda_1, \dots, \lambda_n$?

3. Reorder $(v_1, \dots, v_n)$ so that $(v_1, \dots, v_m)$ are the eigenvectors of T^*T with nonzero eigenavalues. Prove that $\{Tv_1, \dots, Tv_k\}$ is linearly independent. (hint: apply T^*)

4. Prove that $T^*Tv = 0$ implies that $Tv = 0$.

5. Prove that $(Tv_1, \dots, Tv_k)$ (subscript k!) forms a basis of $\mathrm{im}(T)$.

6. Let

   $$w_i := \frac{Tv_i}{\|Tv_i\|},\ 1 \le i \le k$$

   and extend $(w_1, \dots, w_k)$ to an orthonormal basis of W. Show that

   $$\|Tv_i\| = \sqrt{\lambda_i}$$

   and

   $$T(v) = \sum_{i=1}^{\min(m,n)} \sqrt{\lambda_i} \langle v, v_i \rangle w_i.$$