Math 110

July 23, 2018

Positive Semidefinite Operators and Isometries (SOLUTIONS)

1. SA, N, PS

   SA, N, PS, PD, I

   SA, N, PS, PD

   SA, N

   SA, N, I

   N, I

   none

   none

   SA, N, I

   N, I

   SA, N, PS

2.

   N, I

   N, I

   N, I

   SA, N, I

   N

   N, I

   N

   N, I

3. If $\langle Tv, v\rangle \geq 0$ for all v and $\langle Sv, v\rangle \geq 0$ for all v then $\langle (T+S)v, v\rangle = \langle Tv, v\rangle = \langle Sv, v\rangle \geq 0$ so $T+S$ is positive semidefinite.

   To get the proof for positive definite replace $\geq$ by $>$.

   Arbitrary linear combinations of positive semidefinite operators are not positive semidefinite. For example if T is pos. semidef. then $T - T = 0$ is not.

4. Since $S^* = S^{-1}$, then $SS^* = \mathrm{id}_V = S^*S$ so S is normal. By the spectral theorem for complex normal operators, S has an orthonormal eigenbasis. If v is an eigenvector for S with eigenvalue λ, then since S is normal v is an eigenvector for S^* with eigenvalue $\overline{\lambda}$. But v is also an eigenvector for S^{-1} with eigenvalue λ^{-1}. Therefore $|\lambda|^2 = \lambda\overline{\lambda} = \lambda\lambda^{-1} = 1$.

   The first posted version of this problem mistakenly omitted the hypothesis that V be complex. This was a mistake and this problem was not intended to be a trick question. It is not true that S is diagonaliazable if V is real. For example, see a rotation matrix in $\mathbb{R}^2$.

5. S is an isometry if and only it has an *orthonormal* eigenbasis. If it does not have an orthonormal eigenbasis then, no matter what the eigenvalues are, it won't be normal and hence won't be an isometry. Examples of matrices S whose eigenvectors are 1 and -1 but whose eigenvectors are not orthogonal are easy to construct: simply conjugate

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

by a matrix whose columns are not orthogonal.