Math 110

July 23, 2018

Positive Semidefinite Operators and Isometries

1. Consider $\mathbb{R}^2$ with the dot product. For each matrix below, indicate which of the following adjectives apply: self-adjoint, normal, positive semidefinite, positive definite, isometry.

$$\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}3 & 4\\ 4 & 2\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

$$\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}.$$

2. Consider $\mathbb{C}^2$ with the standard inner product,

$$\left\langle \begin{pmatrix}z_1\\ z_2\end{pmatrix}, \begin{pmatrix}w_1\\ w_2\end{pmatrix} \right\rangle := z_1\overline{w_1} + z_2\overline{w_2}.$$

For each matrix below, indicate which of the following adjectives apply: self-adjoint, normal, positive semidefinite, positive definite, isometry.

$$\begin{pmatrix}i & 0\\ 0 & i\end{pmatrix}, \begin{pmatrix}0 & i\\ i & 0\end{pmatrix} \begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}i & -i\\ 1 & 1\end{pmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}i & -i\\ 1 & 1\end{pmatrix}, \begin{pmatrix}1+i & 0\\ 0 & 1-i\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1+i & 0\\ 0 & 1-i\end{pmatrix}.$$

3. (Edited later: part of this problem was false as stated.) Prove that the sum of two positive semidefinite operators is positive semidefinite. Is an arbitrary linear combination of positive semidefinite operators positive semidefinite?

4. (Correction: V needs to be a *complex* vector space.) Let V be finite-dimensional inner product space. Let $S : V \to V$ be an isometry. Prove that S has an orthonormal eigenbasis $(v_1, \ldots, v_n)$ with eigenvalues λ_i satisfying $|\lambda_i| = 1$.

5. Let V be a finite-dimensional inner product space. Is it true that a diagonalizable linear map $S : V \to V$ with eigenvalues λ such that $|\lambda| = 1$ is an isometry?