Math 110
July 18, 2018
Adjoints of Operators (SOLUTIONS)

1. (a) $(M^\top M)_{ik} = \sum_j M^\top_{ij} M_{jk} = \sum_j M_{ji} M_{jk}$ and this is the dot product of the ith column of M with the kth column of M. Therefore it's 1 if $i = k$ and 0 otherwise. Therefore $M^\top M$ is the identity matrix.

   (b) M is a square matrix with orthonormal, hence linearly independent, columns so M is invertible. Since $M^\top M = I$ then $M^\top M M^{-1} = M^{-1}$ so $M^\top = M^{-1}$ so $MM^\top = I$. Reversing the argument in part (a) then shows that $M^\top$ has orthonormal columns, hence M has orthonormal rows.

2. U is invariant for T if and only if $Tu \in U$ for all $u \in U$ if and only if $\langle Tu, w\rangle = 0$ for all $u \in U$ and for all $w \in U^\perp$ if and only if $\langle u, T^* w\rangle = 0$ for all $u \in U$ and $w \in U^\perp$ if and only if $T^* w \in U^\perp$ for all $w \in U^\perp$ if and only if $U^\perp$ is invariant for T^*.

3. Recall rank nullity: $\dim(V) = \dim(\ker(T^*)) + \dim(\operatorname{im}(T^*))$ and the fact that $\ker(T) = \operatorname{im}(T^*)^\perp$. Then

   T is injective if and only if $\ker(T) = \{0\}$ if and only if $\operatorname{im}(T^*)^\perp = \{0\}$ if and only if $\operatorname{im}(T^*) = V$ if and only if $\dim(\ker(T^*)) = 0$ if and only if $\dim(\operatorname{im}(T^*)) = \dim(V)$ if and only if $\operatorname{im}(T^*) = V$ if and only if T^* is surjective.

   To prove that T is surjective if and only if T^* is injective, replace T by T^* in the above and use the fact that $(T^*)^* = T$.

4. Consider $\mathbb{R}^2$ with the dot product and

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

5. This is integration by parts:

$$\langle Tf, g\rangle = \int_{-\infty}^{\infty} i\frac{\partial f}{\partial x}\overline{g}dx = if\overline{g}|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} if\frac{\partial \overline{g}}{\partial x}dx$$

$$= \int_{-\infty}^{\infty} f\overline{i\frac{\partial g}{\partial x}} = \langle f, Tg\rangle$$

   where here I used the fact that the boundary term vanishes since f and g go to 0 as at infinity, and the fact that $\overline{i} = -i$ and the fact that $\overline{a}\overline{b} = \overline{ab}$.

6. Note that V is a complex inner product space. In class we showed that if $\langle Tv, v\rangle = 0$ for all $v \in V$, then $T = 0$. A corollary of this is that

if $\langle Tv, v\rangle = \langle Sv, v\rangle$ then $\langle (T-S)v, v\rangle = 0$ so $T - S = 0$ so $T = S$. Therefore it is enough to show that

$$\langle Tv, v\rangle = \langle T^* v, v\rangle$$

for all $v \in V$. This is immediate since $\langle Tv, v\rangle$ is real:

$$\langle Tv, v\rangle = \overline{\langle Tv, v\rangle} = \langle v, Tv\rangle = \langle T^* v, v\rangle.$$

7. Any eigenvalue of T must be 0 or 1:

$$Tv = \lambda v \Rightarrow T^9 v = \lambda^9 v \text{ and } T^8 v = \lambda^8 v$$

$$\Rightarrow \lambda^9 = \lambda^8 \Rightarrow \lambda = 0, 1.$$

Since T is self-adjoint it is diagonalizable, so by picking an eigenbasis you can represent it by a diagonal matrix M with 0s and 1s on the diagonal. Then matrix satisfies $M^2 = M$ so T satisfies $T^2 = T$.