Math 110

July 18, 2018

Adjoints of Operators

1. For a matrix M, let $M^\top$ denote the tranpose of M: $M^\top_{ij} = M_{ji}$. Let M be a real matrix whose columns are orthonormal with respect to the dot product.

   (a) Show that $M^\top M = I$ where I is the identity matrix.

   (b) (Added later: I forgot to mention that M needs to be square!) Show that M also has orthonormal rows (hint: if $M^\top M = I$ then $MM^\top = I$).

2. Prove that U is an invariant subspace for T if and only if $U^\perp$ is an invariant subspace for T^*.

3. Prove that T is injective if and only if T^* is surjective. Prove that T is surjective if and only if T^* is injective.

4. Give an example of a transformation of an inner product space that is normal but not self-adjoint.

5. Let V be the vector space of infinitely differentiable functions $f : \mathbb{R} \to \mathbb{C}$ such that f and all its derivatives to 0 at ∞ faster than any polynomial. Give V the L^2 inner product:

   $$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx.$$

   Show that the map $T : V \to V$ defined by

   $$f \mapsto i\frac{\partial f}{\partial x}$$

   satisfies

   $$\langle Tf, g \rangle = \langle f, Tg \rangle$$

   for all $f, g \in V$.

6. Let V be a complex inner product space. Prove that $T : V \to V$ is self-adjoint if and only if is $\langle Tv, v \rangle$ is real for all $v \in V$.

7. (Axler 7.B 7) Suppose V is a complex inner product space and $T : V \to V$ is normal and $T^9 = T^8$. Prove that T is self-adjoint and $T^2 = T$. (Hint: what are the eigenvalues of T?)