Math 110
July 16, 2018
Orthogonal Projections (SOLUTIONS)

1. Since $V = U \oplus U^\perp$, write $v = u + w$ where $u \in U$ and $w \in U^\perp$. By definition $P_U(v) = u$. Since $(U^\perp)^\perp = U$, then $P_{U^\perp}(v) = w$. Therefore $P_U + P_{U^\perp} = \mathrm{id}_V$.

2. Let $u_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$. Then the projection of e_1 onto U is
$$(e_1 \cdot u_1)u_1 = \frac{1}{2}(e_1 + e_2)$$
and the projection of e_2 onto U is the same thing:
$$(e_2 \cdot u_1)u_1 = \frac{1}{2}(e_1 + e_2).$$
Since the columns of the matrix for a P_U are $P_U(e_1)$ and $P_U(e_2)$, the matrix for P_U is
$$\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}.$$

3. Let
$$u_3 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}.$$
Then u_3 is an orthonormal basis for $U^\perp$. Similarly to the last question,
$$P_{U^\perp}(e_i) = (e_i \cdot u_3)u_3 = \frac{1}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}.$$
Therefore the matrix for $P_{U^\perp}$ is
$$\frac{1}{3}\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}.$$
By question 1 the matrix for P_U is
$$\begin{pmatrix}1 & & \\ & 1 & \\ & & 1\end{pmatrix} - \frac{1}{3}\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix} = \begin{pmatrix}\frac{2}{3} & -\frac{1}{3} & -\frac{1}{3}\\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3}\\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3}\end{pmatrix}.$$

4. (a) The bilinearity properties follow from the fact that A is linear and the dot product is bilinear. The symmetry property follows from the fact that A is symmetric. Write
$$x = \begin{pmatrix}x_1\\x_2\end{pmatrix}$$

so that

$$\langle x, x\rangle = 4x_1^2 - 4x_1x_2 + 5x_2^2.$$

Complete the square to see that this is a sum of two positive things:

$$4x_1^2 - 4x_1x_2 + 5x_2^2 = (2x_1 - x_2)^2 - x_2^2 + 5x_2^2 = (2x_1 - x_2)^2 + 4x_2^2.$$

Therefore $\langle x, x\rangle \geq 0$ for all x and if $\langle x, x\rangle = 0$, then $2x_1 - x_2 = 0$ and $x_2 = 0$ so that $x_1 = x_2 = 0$.

(b) The sets $\|x\|^2 = c$ are of the form

$$4x_1^2 - 4x_1x_2 + 5x_2^2 = c$$

that is,

$$(2x_1 - x_2)^2 + 4x_2^2 = c.$$

This is a conic section and is bounded (doesn't go to infinity) since $4x_2^2 < c$ and $(2x_1 - x_2)^2 < c$. Therefore it's an ellipse, though the equation might not look so familiar since it's a rotated ellipse.

The following might help to see what's going on. Suppose that you can find a symmetric matrix B such that $A = B^2$. Then $B = B^*$ and so

$$x \cdot Ax = x \cdot B^2x = Bx \cdot Bx.$$

Therefore the solutions of to the equation

$$x \cdot Ax = c$$

are the same as the those of the equation

$$Bx \cdot Bx = c$$

which are the same as $B^{-1}y$ for

$$y \cdot y = c.$$

The solutions to the last equation are just circles, and the points $B^{-1}y$ are circles stretched by the linear map B^{-1}. Since A is symmetric with nonnegative eigenvalues, it turns out you can find B such that $B^2 = A$. The easy way is to diagonalize A:

$$A = P \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} P^{-1}$$

for some $\lambda_1, \lambda_2 > 0$. Then define

$$B = P \begin{pmatrix} \sqrt{\lambda_1} & \\ & \sqrt{\lambda_2} \end{pmatrix} P^{-1}.$$

We'll cover square roots more later.

(c) $\|e_1\| = 2$ so $u_1 = \frac{1}{2}e_1$. Then

$$\langle e_2, u_1 \rangle = \frac{1}{2} e_2 \cdot A e_1 = -1$$

so

$$e_2 - \langle e_2, u_1 \rangle u_1 = e_2 + \frac{1}{2} e_1$$

so

$$u_2 = \frac{e_2 + \frac{1}{2}e_1}{2} = \frac{1}{4} e_1 + \frac{1}{2} e_2.$$

5. The vector $x = (1, 0)$ would be a nonzero vector such that $\langle x, x \rangle = 0$.

6. The idea is to move P to the origin in a way that preserves distances, perform an orthogonal projection, and then move P back. Namely, let u be such that $P + u$ passes through 0. Let w be the closest point to $v + u$ on $P + u$. Then $w - u$ is the closest point to v on P. Therefore the point on P closest to v is $P_{P+u}(v + u) - u$.

7. Let

$$x = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 1 \\ 3 \\ 2 \\ 2 \\ 4 \\ 4 \end{pmatrix}, \quad y = \begin{pmatrix} 2 \\ 7 \\ 5 \\ 2 \\ 5 \\ 6 \\ 3 \\ 8 \\ 7 \end{pmatrix}.$$

Orthogonally project y onto $\mathrm{Span}(x)$:

$$P_{\mathrm{Span}(x)}(y) = ax.$$

The number a is the number that minimizes

$$\sum_{i=1}^{9} (ax_i - y_i)^2$$

and therefore is the constant in the desired linear relation. Normalize x to get u, an orthonormal basis for $\mathrm{Span}(x)$:

$$u = \frac{1}{8} \begin{pmatrix} 1 \\ 3 \\ 2 \\ 1 \\ 3 \\ 2 \\ 2 \\ 4 \\ 4 \end{pmatrix}$$

$$P_{\text{Span}(x)}(y) = (y \cdot u)u.$$

This turns out to be $2x$, so a indeed is 2 exactly.

8. Let v_i be the ith column of M so that M can be represented by the list of vectors $(v_1, \ldots, v_n)$ viewed as column vectors placed next to each other.

   Perform Gram-Schmidt on $(v_1, \ldots, v_n)$ to get $(u_1, \ldots, u_n)$. Then v_k is in the span of $(u_1, \ldots, u_k)$ so that

   $$v_k = (v_i \cdot u_1)u_1 + (v_i \cdot u_2)u_2 + \cdots + (v_k \cdot u_k)u_k.$$

   Therefore

   $$(v_1, \ldots, v_n) = (u_1, \ldots, u_n)R$$

   where the kth column of R is

   $$\begin{pmatrix} v_k \cdot u_1 \\ v_k \cdot u_2 \\ \vdots \\ v_k \cdot u_k \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

   Let Q be the matrix whose ith column is u_i. Then $M = QR$.