Math 110

July 16, 2018

Orthogonal Projections

1. Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space and let P_U be the orthogonal projection onto the subspace U. Show that $P_{U^\perp} = \mathrm{id}_V - P_U$.

2. Consider $\mathbb{R}^2$ with the dot product. Let U be the span of $(1, 1)$. With respect to the standard basis vectors, what is the matrix for the orthogonal projection onto U?

3. Consider $\mathbb{R}^3$ with the dot product. Let U be the subspace of vectors (x, y, z) such that $x + y + z = 0$. With respect to the standard basis vectors, what is the matrix for the orthogonal projection onto U?

4. Let
$$A = \begin{pmatrix} 4 & -2 \\ -2 & 5 \end{pmatrix}.$$
Put an inner product on $\mathbb{R}^2$ by $\langle x, y \rangle := x \cdot Ay$.
   (a) Check that $\langle \cdot, \cdot \rangle$ is an inner product.
   (b) What are the sets $\|x\|^2 = $ constant? What do they look like in the plane, geometrically?
   (c) Start with the standard basis (e_1, e_2) and do Gram-Schmidt to get an orthonormal basis. (In particular, note that (e_1, e_2) is not orthonormal for $\langle \cdot, \cdot \rangle$.)

5. In the last problem, what would "go wrong" if you used $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ instead?

6. Consider $\mathbb{R}^3$ with the dot product. Let v be a vector in $\mathbb{R}^3$ and P a plane possibly not passing through the origin. How could you use an orthogonal projection to the find the point on P closest to v?

7. In a highly unscientific study, a GSI collects information from nine students about how much time they studied for a quiz and compares it to their scores on that quiz (out of 8):

Student	Hours Studied	Quiz Score
1	1	2
2	3	7
3	2	5
4	1	2
5	3	5
6	2	6
7	2	3
8	4	8
9	4	7

The GSI guesses that there's a linear relation $f(x) = ax$ writing the quiz score as a function of the number of hours studied x. It seems like a should be around 2 but use the available data and orthogonal projection to get the best guess for a.

8. Consider the dot product on $\mathbb{R}^n$. Using Gram-Schmidt on the columns of a real 2×2 matrix M, should that M can be written as a product $M = QR$ where Q has orthonormal columns and R is upper triangular (that is, R has zeroes below the diagonal). Repeat this process for a 3×3 real matrix. At this point you should be able to convince yourself that this is true for arbitrary $n \times n$ real matrices.

   This fact is used in several computer algorithms that find eigenvalues of matrices.