Math 110

July 12, 2018

Orthogonal Complement (SOLUTIONS)

1. (a)

$$U^\perp = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} |a+b+2c=0 \text{ and } 2a+b+2c=0 \right\}.$$

Solving the system of two equations

$$a+b+2c=0$$

$$2a+b+2c=0$$

shows that the solution is $a=0$, $b=-2c$ so that $U^\perp$ consists of vectors of the form

$$\begin{pmatrix} 0 \\ -2c \\ c \end{pmatrix}.$$

A basis is

$$\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}.$$

(b) Gram-Schmidt tells us to normalize the first (and, in this case, only) basis vector. Thus

$$\frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$

is an orthonormal basis for $U^\perp$.

(c) Recall that if v is a vector in an inner product space and U a subspace then the closest point on U to v is the orthogonal projection of v to U.

Let P_U be the orthogonal projection onto U.

Let u_1, u_2, u_3 be an orthnormal basis of $\mathbb{R}^3$ such that u_1 and u_2 form an orthonormal basis of U. Then since

$$v = (v \cdot u_1)u_1 + (v \cdot u_2)u_2 + (v \cdot u_3)u_3$$

the projection of v onto U is

$$P_U(v) = (v \cdot u_1)u_1 + (v \cdot u_2)u_2.$$

or

$$P_U(v) = v - (v \cdot u_3)u_3.$$

If

$$v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

then one can use u_3 from part (b) to compute the projection of v onto U:

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \right) \frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3/5 \\ 6/5 \end{pmatrix}.$$

Alternatively, one could apply Gram-Schmidt to

$$\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

to an orthonormal basis (u_1, u_2) for U:

$$\frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \frac{1}{\sqrt{30}} \begin{pmatrix} 5 \\ -1 \\ -2 \end{pmatrix}.$$

And then, using the formula

$$P_U(v) = (v \cdot u_1)u_1 + (v \cdot u_2)u_2$$

for $v = (1, 1, 1)$,

$$P_U \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \frac{1}{6} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right) \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{30}} \begin{pmatrix} 5 \\ -1 \\ -2 \end{pmatrix} \right) \frac{1}{\sqrt{30}} \begin{pmatrix} 5 \\ -1 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 3/5 \\ 6/5 \end{pmatrix}.$$

2. (a)

$$U = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} | a + b + 2c = 0 \right\}$$

and thus consists of vectors of the form

$$\begin{pmatrix} -b - 2c \\ b \\ c \end{pmatrix}$$

a basis of which is

$$v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}.$$

(b) To do Gram-Schmidt:

$$u_1 = \frac{1}{\|v_1\|} v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$v_2 - (v_2 \cdot u_1) u_1 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} - \left(\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right) \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

so that

$$u_2 = \frac{v_2 - (v_2 \cdot u_1) u_1}{\|v_2 - (v_2 \cdot u_1) u_1\|} = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

Therefore an orthonormal basis is

$$\frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

(c) Let u_3 be an orthormal basis for U:

$$u_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

Then the closest point to $(1, 1, 1)$ on U is

$$P_U \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right) \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 2/3 \\ 4/3 \end{pmatrix}.$$