Math 110

July 11, 2018

Inner Product Spaces and Gram-Schmidt (SOLUTIONS)

1. Suppose $\langle u, v\rangle = 0$. Then

$$\|u+v\|^2 = \langle u+v, u+v\rangle = \langle u, v\rangle + \langle u, v\rangle + \langle v, u\rangle + \langle v, v\rangle = \|u\|^2 + \|v\|^2.$$

2.

$$\|u+v\|^2+\|u-v\|^2 = \|u\|^2+\langle u, v\rangle+\langle v, u\rangle+\|v\|^2+\|u\|^2-\langle u, v\rangle-\langle v, u\rangle+\|v\|^2.$$

3. (e_1, e_2), $(e_1, -e_2)$, (e_2, e_1), $(-e_2, e_1)$.

4. Suppose that e_1 is the first basis vector. The next basis vector has to be orthogonal to e_1 and thus in the span of e_2 and e_3. Since it has to have unit norm, it can be written

$$\cos(\theta)e_2 + \sin(\theta)e_3$$

for some angle θ. A vector orthogonal to this one (and orthogonal to e_1) must be of the form

$$\pm(-\sin(\theta)e_2 + \cos(\theta)e_3).$$

Therefore the possible bases with e_1 first are

$$(e_1, \cos(\theta)e_2 + \sin(\theta)e_3, \pm(-\sin(\theta)e_2 + \cos(\theta)e_3))$$

(two possibilities for every angle θ). There are two other analogous cases where e_1 is the second or third basis vector.

5. Let $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Then

$$u_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

$$v_2 - \langle v_2, u_1\rangle u_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \left(\begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)\frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 6/5 \\ -3/5 \end{pmatrix}$$

and

$$u_2 = \frac{v_2 - \langle v_2, u_1\rangle u_1}{\|v_2 - \langle v_2, u_1\rangle u_1\|} = \frac{\sqrt{5}}{3}\begin{pmatrix} 6/5 \\ -3/5 \end{pmatrix}.$$

u_1, u_2 is an orthonormal basis.

6. Let
$$v_1 = \begin{pmatrix}1\\1\\1\end{pmatrix},\ v_2 = \begin{pmatrix}1\\1\\-2\end{pmatrix}.$$
Then
$$u_1 = \frac{1}{\|v_1\|}v_1 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}.$$
$$\langle v_2, u_1\rangle = 0 \Rightarrow u_2 = \frac{1}{\|v_2\|}v_2 = \frac{1}{2\sqrt{2}}\begin{pmatrix}1\\1\\-2\end{pmatrix}.$$
The result of Gram-Schmidt is the orthonormal basis (u_1, u_2).

The orthogonal complement is the set of vectors (a, b, c) such that
$$a + b + c = 0$$
$$a + b - 2c = 0$$
i.e.,
$$c = 0,\ a + b = 0.$$
These vectors are of the form $(a, -a, 0)$ and so a basis is $(1, -1, 0)$. Normalizing this vector gives
$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}.$$
This is an orthonormal basis.

7. Recall that the orthogonal projection of a vector w onto the span of another vector v is
$$P_{\mathrm{Span}(v)}(w) = \frac{\langle w, v\rangle}{\langle v, v\rangle}v.$$
This formula works for any inner product space. For this problem, the inner product is the L^2 inner product:
$$\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}dx.$$
Note that for real numbers a that $\overline{a} = a$.

Hence the projection of the function $f(x) = x$ onto the span of the constant function 1 is
$$\frac{\langle f, 1\rangle}{\langle 1, 1\rangle}1 = \frac{\int_0^1 x dx}{\int_0^1 1^2 dx}1 = \frac{1}{2}.$$

Then $x - 1/2$ is orthogonal to a the constant functions and

$$f = \underbrace{\frac{1}{2}}_{f_1} + \underbrace{x - \frac{1}{2}}_{f_2}$$

is the desired decomposition of f.

Similarly, for $g(x) = \cos(2\pi x)$, the projection of g onto the span of the constant function 1 is

$$\frac{\langle g, 1\rangle}{\langle 1, 1\rangle} 1 = \frac{\int_0^1 \cos(2\pi x) dx}{\int_0^1 1^2 dx} 1 = 0.$$

Therefore g is already orthogonal to the constant function 1. Hence

$$g = \underbrace{0}_{g_1} + \underbrace{\cos(2\pi x)}_{g_2}$$

is the desired decomposition of g.

8. Let $u_1, u_2, \ldots, u_n$ be an orthonormal basis of V. For an arbitrary vector v,

$$v = \langle v, u_1\rangle u_1 + \cdots + \langle v, u_n\rangle u_n$$

Therefore

$$\phi(v) = \langle v, u_1\rangle \phi(u_1) + \cdots + \langle v, u_n\rangle \phi(u_n) = \sum_{i=1}^{n} \langle v, \overline{\phi(u_i)} u_i\rangle.$$

Therefore $u = \sum_{i=1}^{n} \overline{\phi(u_i)} u_i$ is such that

$$\phi(v) = \langle v, u\rangle$$

for all $v \in V$.

9. The columns of an "orthogonal" matrix are an orthonormal set, so why isn't it called an "orthonormal" matrix instead?