Math 110

July 11, 2018

Inner Product Spaces and Gram-Schmidt

1. Prove the "Pythagorean theorem" for inner product spaces: if u and v are orthogonal, then $\|u+v\|^2 = \|u\|^2 + \|v\|^2$. How does this relate to the actual Pythagorean theorem?

2. Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. Prove the parallelogram equality: for $u, v \in V$:

$$\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2)$$

   (I did this in class for the dot product, but the proof is the same in both cases.)

3. Consider $\mathbb{R}^2$ with its usual dot product. How many orthonormal bases of $\mathbb{R}^2$ include the vector $(1,0)$? Write them all down.

4. Consider $\mathbb{R}^3$ with its usual dot product. How many orthonormal bases of $\mathbb{R}^3$ include the vector $(1,0,0)$? Write them all down.

5. Consider $\mathbb{R}^2$ with its usual dot product. Apply Gram-Schmidt to the vectors

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

   to get an orthonormal basis of $\mathbb{R}^2$.

6. Consider $\mathbb{R}^3$ with its usual dot product. Apply Gram-Schmidt to the vectors

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

   to get an orthonormal basis for their span. What is a basis for the orthogonal complement to their span?

7. Let V be the vector space of continuous functions on the interval $[0,1]$ with the L^2 inner product. Let U be the span of the contant function 1. Write the function $f(x) = x$ as $f = f_1 + f_2$ where $f_1 \in U$ and $f_2 \in U^\perp$. Write the function $g(x) = \cos(2\pi x)$ as $f = g_1 + g_2$ where $g_1 \in U$ and $g_2 \in U^\perp$.

8. Let $(V, \langle \cdot, \cdot \rangle)$ be a finite-dimensional inner product space over $\mathbb{F}$. Let $\phi : V \to \mathbb{F}$ be a linear map. Prove that there exists $u \in V$ such that $\langle v, u \rangle = \phi(v)$ for all v. (Hint: a linear map is defined by its restriction to a basis. Consider an orthonormal basis of V.)

9. Consider $\mathbb{R}^n$ with the usual dot product. An $n \times n$ matrix whose columns form an orthonormal basis of $\mathbb{R}^n$ is called an "orthogonal matrix". Explain why this term is (unfortunately) unsatisfying.