Math 110
July 2, 2018
The Proof of Jordan Normal Form

Remember that not every linear transformation over $\mathbb{C}$ is diagonalizable. The following theorem, originally proved by Camille Jordan in the 19th century, is the "next best thing". Even if you cannot diagonalize a transformation, you can find a basis where it is represented by a block diagonal matrix and the blocks themselves are almost diagonal:

Theorem 1. *Let $T : V \to V$ be linear map and V a finite dimensional complex vector space. Then there exist indecomposable invariant subspaces $U_1, \dots, U_k$, numbers $\lambda_1, \dots, \lambda_k \in \mathbb{C}$, and bases $B_1, \dots, B_k$ (where B_i is a basis of U_i) such that $V = U_1 \oplus \cdots \oplus U_k$ and, with respect to the basis B_i, $T|_{U_i}$ can be represented by a matrix that looks like*

$$\begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ 0 & 0 & \lambda_i & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & \lambda_i \end{pmatrix}$$

(a matrix with the same entry λ_i on the main diagonal and 1s on the diagonal above that).

Since V can be decomposed into a direct sum of indecomposable subspaces, it is enough to prove the theorem on each of those subspaces separately:

Proposition 2. *Let $T : V \to V$ with V a finite dimensional complex vector space and suppose that V is an indecomposable invariant subspace. Then there exists a basis $(v_1, \dots, v_n)$ for V and a number λ such that with respect this basis T is represented by a matrix of the form*

$$\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ 0 & 0 & \lambda & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}$$

The point of this handout is to provide steps to prove Proposition 2. The exercises come in two parts.

1 Iterations of a linear map T

Exercise 3. *(easy) Let $T : V \to W$ be an isomorphism and let $U \subset V$ be a subspace. Show that T maps U invertibly onto its image.*

Exercise 4. *Let $T : V \to W$ be linear surjection such that $\dim(V) = \dim(W)$. Show that T is invertible.*

Exercise 5. *Let $T : V \to V$ be a linear map. Suppose that $\mathrm{im}(T^k) = \mathrm{im}(T^{k+1})$ for some k. Show that $\mathrm{im}(T^\ell) = \mathrm{im}(T^k)$ for all $\ell \geq k$. (hint: consider the sequence of linear surjections $\mathrm{im}(T^k) \xrightarrow{T} \mathrm{im}(T^{k+1}) \xrightarrow{T} \mathrm{im}(T^{k+2}) \xrightarrow{T} \mathrm{im}(T^{k+3}) \xrightarrow{T} \cdots$)*

Exercise 6. *Let $T : V \to V$ with $\dim(V) = n$. Show that $\mathrm{im}(T^n) = \mathrm{im}(T^{n+1}) = \mathrm{im}(T^{n+2}) = \cdots$.*

Exercise 7. *Let $T : V \to V$ with $\dim(V) = n$. Show that $\ker(T^n) \cap \mathrm{im}(T^n) = \{0\}$. Then show that $V = \ker(T^n) \oplus \mathrm{im}(T^n)$.*

2 Nilpotent Linear Maps

Definition 8. A linear map $N : V \to V$ is called nilpotent if $N^n = 0$ for some n.

Exercise 9. *Let $T : V \to W$ and let X be a linearly independent subset of V such that $T(X)$ is a basis of W. Show that $V = \mathrm{Span}(X) \oplus \ker(T)$.*

The next three exercises are successive iterations of the same idea:

Exercise 10. *Suppose $N : V \to V$ is such that $N \neq 0$ and $N^2 = 0$. Show that there exist disjoint linearly independent ordered sets $X(0) \subset V$ and $X(1) \subset V$ such that $N(X(1))$ is a basis of $\mathrm{im}(N)$ and $X(1) \cup N(X(1)) \cup X(0)$ is a basis of V.*

A potentially helpful diagram is:

$$\begin{array}{ccccc} V & \xrightarrow{\quad N \quad} & \mathrm{im}(N) & \xrightarrow{\quad N \quad} & 0 \\ X(1) & & N(X(1)) & & \\ N(X(1)) & & & & \\ X(0) & & & & \end{array}$$

The top line contains three vector spaces. Below each vector space is a collection of finite subsets that together form a basis for that space. A subset maps to the subset to the right of it. If there is no subset to the right of it, it maps to 0.

Exercise 11. *Suppose that $N : V \to V$ is such that $N^2 \neq 0$ and $N^3 = 0$. Show that there exist disjoint linearly independent ordered sets $X(0) \subset V$, $X(1) \subset V$, $X(2) \subset V$ such that*

$$X(0) \cup X(1) \cup N(X(1)) \cup X(2) \cup N(X(2)) \cup N^2(X(2))$$

is a basis of V.

As in the previous exercise, a potentially helpful picture is

$$\begin{array}{ccccccc}
V & \xrightarrow{N} & \mathrm{im}(N) & \xrightarrow{N} & \mathrm{im}(N^2) & \xrightarrow{N} & 0 \\
X(2) & & N(X(2)) & & N^2(X(2)) & & \\
N(X(2)) & & N^2(X(2)) & & & & \\
X(1) & & N(X(1)) & & & & \\
N^2(X(2)) & & & & & & \\
N(X(1)) & & & & & & \\
X(0) & & & & & &
\end{array}$$

Exercise 12. *Suppose that $N : V \to V$ is such that $N^3 \neq 0$ and $N^4 = 0$. Show that there exist disjoint linearly independent ordered sets $X(j) \subset V$, $j = 0, 1, 2, 3$ such that*

$$\bigcup_{j=0}^{3} \bigcup_{i=0}^{j} N^i(X(j))$$

is a basis of V.

At this point you should be able to convince yourself that the following could be proved by induction (though I don't ask you to do so):

Proposition 13. *Suppose that $N : V \to V$ is such that $N^k \neq 0$ and $N^{k+1} = 0$. Show that there exist disjoint linearly independent ordered sets $X(j) \subset V$, $j \in \{0, \ldots, k\}$ such that*

$$\bigcup_{j=0}^{k} \bigcup_{i=0}^{j} N^i(X(j))$$

is a basis of V.

From now on assume the truth of Proposition 13.

Exercise 14. *Let $N : V \to V$ be nilpotent. Show that there exists a basis of V which is a union of lists of the form $(v, N(v), N^2(v), \ldots, N^j(v))$.*

Exercise 15. *Let $N : V \to V$ be nilpotent and such that V is an indecomposable subspace. Show that there exists a basis of V of the form*

$$(N^{n-1}(v), N^{n-2}(v), \ldots, N(v), v).$$

Exercise 16. *Show that, with respect to the basis in the previous exercise, the matrix for N is of the form*

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

That is, with 1s on the diagonal above the main diagonal and zeroes everywhere else.

3 Conclusion of the Proof

Exercise 17. *Let V be a complex vector space of dimension n. Let $T : V \to V$ be a linear map for which V is an indecomposable subspace for T. Let λ be an eigenvalue for T. Use Exercise 7, applied to $T - \lambda\,\mathrm{id}_V$, to show that $V = \ker(T - \lambda\,\mathrm{id}_V)^n$ so that $T - \lambda\,\mathrm{id}_V$ is nilpotent.*

Exercise 18. *Combine Exercise 16 and Exercise 17 to prove Proposition 2.*