Math 110
June 28, 2018
More Eigenstuff (SOLUTIONS)

1. Let $\begin{pmatrix} a \\ b \end{pmatrix}$ be an eigenvector with eigenvalue λ. Then

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} a\cos(\theta) - b\sin(\theta) \\ a\sin(\theta) + b\cos(\theta) \end{pmatrix} = \begin{pmatrix} \lambda a \\ \lambda b \end{pmatrix}$$
$$\Rightarrow b = -\frac{a\sin(\theta)}{\cos(\theta) - \lambda}$$
$$\Rightarrow a(\cos(\theta) - \lambda) + \frac{a(\sin(\theta))^2}{\cos(\theta) - \lambda} = 0$$
$$\Rightarrow (\cos(\theta) - \lambda)^2 + (\sin(\theta))^2 = 0$$
$$\Rightarrow \lambda^2 - 2\cos(\theta) + 1 = 0$$
$$\Rightarrow \lambda = \cos(\theta) \pm i\sin(\theta) = e^{\pm i\theta}.$$

The eigenvectors are of the form (for $a \neq 0$):

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ -\frac{a\sin(\theta)}{\cos(\theta) - (\cos(\theta) \pm i\sin(\theta))} \end{pmatrix} = \begin{pmatrix} a \\ \mp ia \end{pmatrix}.$$

For concreteness, set $a = 1$ to get two eigenvectors:

$$\begin{pmatrix} 1 \\ \mp i \end{pmatrix}.$$

2. In order to show that three statements A, B, C are equivalent it is enough to show that $A \Rightarrow B \Rightarrow C \Rightarrow A$.

Suppose that $P^2 = P$. If λ is an eigenvalue of P then $Pv = \lambda v$ so $P^2 v = \lambda Pv \Rightarrow Pv = \lambda Pv$ so $\lambda = 0$ or 1. I claim that $V = \ker(P) \oplus \operatorname{im}(P)$. Let $v \in \ker(P) \cap \operatorname{im}(P)$. Then $Pv = 0$ and $v = Pw$ for some $w \in V$ so $0 = Pv = P^2 w = Pw = v$. Hence $\ker(P) \cap \operatorname{im}(P) = \{0\}$ and $\ker(P) + \operatorname{im}(P)$ is a direct sum. Hence $\dim(\ker(P) \oplus \operatorname{im}(P)) = \dim \ker(P) + \dim(\operatorname{im}(P))$ which by rank-nullity is $\dim(V)$. Hence $\ker(P) \oplus \operatorname{im}(P)$ is a subspace of V of dimension equal to $\dim(V)$ and so must be all of V. Note that $\operatorname{im}(P)$ is the 1-eigenspace of P and $\ker(P)$ is the 0-eigenspace.

Suppose that P is diagonalizable with eigenvalues belonging to the set $\{0, 1\}$. Let U be the 1-eigenspace and W the 0-eigenspace. Then clearly $U \cap W = \{0\}$ and, since P is diagonalizable, $U + W = V$. Hence $V = U \oplus W$. P acts as the projection onto U.

Suppose that there exist subspaces U, W of V such that $V = U \oplus W$ and P is the projection onto U: $P(u + w) = u$. Then $P^2(u + w) = P(u) = u = P(u + w)$ so $P^2 = P$.

3. False: consider the matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

acting on $\mathbb{R}^2$. As shown in class, the span of e_1 is the only invariant subspace, so in particular V cannot be written as $U \oplus W$ for U and W invariant subspaces.