Math 110
June 27, 2018
Eigenvectors (SOLUTIONS)

1. Let $u \in U_1 \cap U_2$. Since $u \in U_1$ then $T(u) \in U_1$. Since $u \in U_2$ then $T(u) \in U_2$. Hence $T(u) \in U_1 \cap U_2$.

2. Consider a linear transformation that is, with respect to a fixed basis, given by a rotation matrix
$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$
for $\theta \notin \pi\mathbb{Z}$. Such a rotation does not fix a line.

3. Let V be the vector space of polynomials in the variable x with a complex coefficients. Let $T(p) = xp$. Then $Tp \neq \lambda p$ for any nonzero p because T increases the degree of the polynomial.

4. This proof uses the solution to 7. Let $T(u) = \lambda_1 u$, $T(v) = \lambda_2 v$ and $T(u+v) = \lambda_3(u+v)$. Then $T(u+v) = \lambda_1 u + \lambda_2 v$ and
$$\lambda_1 u + \lambda_2 v = \lambda_3(u+v)$$
so
$$(\lambda_1 - \lambda_3)u + (\lambda_2 - \lambda_3)v = 0$$
If $\lambda_1 = \lambda_2$, then $\lambda_3 = \lambda_1 = \lambda_2$. If $\lambda_1 \neq \lambda_2$, then problem 7 implies that u and v are linearly independent. Then $\lambda_1 - \lambda_3 = \lambda_2 - \lambda_3 = 0$, contradicting $\lambda_1 \neq \lambda_2$.

5. Define P by $P(u_i) := v_i$. Then $P^{-1}TPu_i = P^{-1}Tv_i = \lambda_i P^{-1}v_i = \lambda_i u_i$.

6. $T^3 = \mathrm{id}$, so after three iterations it takes the standard basis back to itself. Let $\begin{pmatrix} a \\ b \end{pmatrix}$ be an eigenvector for T. Then
$$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} -b \\ a-b \end{pmatrix} = \begin{pmatrix} \lambda a \\ \lambda b \end{pmatrix}$$
Solving these equations shows that $\lambda^2 + \lambda + 1 = 0$. This equation does not have solutions over $\mathbb{R}$, so T does not have any eigenvectors.

   Viewing T instead as a transformation $\mathbb{C}^2 \to \mathbb{C}^2$, $\lambda^2 + \lambda + 1 = 0$ does have two solutions, and so you can use these solve for the eigenvectors. Let ζ_1 and ζ_2 be the two roots. Explicitly ζ_1 and ζ_2 are $\frac{-1 \pm i\sqrt{3}}{2}$. Then
$$\begin{pmatrix} 1 \\ -\zeta_1 \end{pmatrix} \begin{pmatrix} 1 \\ -\zeta_2 \end{pmatrix}$$

are two eigenvectors. Since these are linearly independent, they form a basis.

7. Suppose that $a_1 v + a_2 w = 0$ is a linear relation. We'd like to show that $a_1 = a_2 = 0$. Suppose that $a_1 \neq 0$. Then

$$v = -\frac{a_2}{a_1} w$$

so v is a scalar multiple of w. Then v must have the same eigenvalue as w since

$$T(cw) = cT(w) = c\lambda w = \lambda c w.$$

Something similar works if $a_2 \neq 0$. Since v and w have different eigenvalues, it follows that both a_1 and a_2 must be 0.

8. The identity transformation, which sends every vector to itself, has the property that every nonzero vector is an eigenvector with eigenvalue 1.

9. Eigenvectors with nonzero eigenvalue are in the image of T. Eigenvectors with different eigenvalues are linearly independent, and there can be no more than k elements in a linearly independent subset of a space of dimension k. Hence if $\dim(\mathrm{Im}(T)) = k$ there can no more than k nonzero eigenvalues. Adding on the possibility of the zero eigenvalue implies that T can have at most $k+1$ eigenvalues.

I didn't prove in class that if $v_1, \ldots, v_n$ are eigenvectors with different eigenvalues then they are linearly independent. Here is a proof, by induction on the number of eigenvectors. Because an eigenvector is nonzero, a set with one eigenvector is linearly independent. For the inductive step, suppose you know that $(n-1)$ eigenvectors with distinct eigenvalues are linearly independent. Let $v_1, \ldots, v_n$ be eigenvectors with different eigenvalues $\lambda_1, \ldots, \lambda_n$. Suppose

$$a_1 v_1 + \cdots + a_n v_n = 0$$

Applying T both sides shows that

$$\lambda_1 a_1 v_1 + \cdots + \lambda_n a_n v_n = 0$$

or instead multiplying by λ_1 shows that

$$\lambda_1 a_1 v_1 + \lambda_1 a_2 v_2 + \lambda_1 a_3 v_3 + \cdots + \lambda_1 a_n v_n = 0.$$

Subtracting these two shows that

$$(\lambda_2 - \lambda_1) a_2 v_2 + (\lambda_3 - \lambda_1) a_3 v_3 + \cdots + (\lambda_n - \lambda_1) a_n v_n = 0.$$

By the inductive hypothesis $v_2, \ldots, v_n$ are linearly independent. Therefore

$$(\lambda_i - \lambda_1) a_i = 0$$

for each $2 \leq i \leq n$. Since $\lambda_i - \lambda_1 \neq 0$, each $a_i = 0$ for $2 \leq i \leq n$. Since $v_1 \neq 0$ it follows that $a_1 = 0$. Hence

$$a_1 v_1 + \cdots + a_n v_n = 0 \Rightarrow a_i = 0 \ \forall i.$$

10. Let $Tv = \lambda v$. Then $S^{-1}TS(S^{-1}v) = S^{-1}Tv = \lambda S^{-1}v$ so that if v is an eigenvector for T with eigenvalue λ then $S^{-1}v$ is an eigenvector for $S^{-1}TS$ with eigenvalue λ. Similarly, if w is an eigenvector for $S^{-1}TS$ with eigenvalue λ then Sw is an eigenvector for T with eigenvalue λ.