Math 110
June 27, 2018
Eigenvectors

1. Let $T : V \to V$ and let U_1 and U_2 be invariant subspaces for V. Show that $U_1 \cap U_2$ is an invariant subspace.

2. Let V be a real vector space with $\dim(V) = 2$. Find a linear map $T : V \to V$ so that the only invariant subspaces of T are $\{0\}$ and V.

3. Find a linear map $T : V \to V$ of a complex vector space to itself such that T has no eigenvectors.

4. Suppose that u, v, and $u + v$ are eigenvectors. Show they have the same eigenvalues.

5. Let $T : V \to V$ be diagonalizable so, in particular, it has a basis of eigenvectors. Let $(v_1, \ldots, v_n)$ be this basis. Let $(u_1, \ldots, u_n)$ be another basis of V. Find an invertible linear transformation P such that $(u_1, \ldots, u_n)$ is an eigenbasis for $P^{-1} \circ T \circ P$.

6. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given (with respect to the standard basis vectors) by
$$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$
If you were to iterate this transformation on the standard basis, what happens? Is T diagonalizable? If so, find a basis of eigenvectors. If you view T instead as a linear map $\mathbb{C}^2 \to \mathbb{C}^2$, is it diagonalizable? If so, find a basis of eigenvectors.

7. Prove that if v_1 and v_2 are eigenvectors of T with different eigenvalues, then $\{v_1, v_2\}$ is linearly independent.

8. Can you find a linear transformation $T : V \to V$ with two linear linearly independent eigenvectors with the same eigenvalues?

9. Suppose $T : V \to V$ has $\dim(\mathrm{im}(T)) = k$. Prove that T has at most $k + 1$ distinct eigenvalues.

10. Show that $S^{-1} \circ T \circ S$ and T have the same eigenvalues. What is the relationship between the eigenvectors of T and the eigenvectors of $S^{-1} \circ T \circ S$?

11. (Challenge) Assume the fundamental theorem of algebra: that if p is a polynomial with complex coefficients then $p(z) = (z - r_1) \cdots (z - r_n)$ for some $r_1, \ldots, r_n \in \mathbb{C}$. Show that if p is a polynomial with real coefficients then p factors as a product of linear and quadratic factors with coefficients in $\mathbb{R}$.