Math 110
June 25, 2018
Linear Maps (SOLUTIONS)

1. `[Removed]`

2. Let $(v_1, \ldots, v_n)$ be a basis of V. Then $\{T(v_1), \ldots, T(v_n)\}$ spans the image of T: any vector $T(v)$ can be written

$$T(v) = T(a_1 v_1 + \cdots + a_n v_n) = a_1 T(v_1) + \cdots + a_n T(v_n).$$

By exercise 2 of the last handout, a spanning set contains a basis, so the number of elements in a basis of $\mathrm{im}(T)$ is less than or equal to n. Hence $\dim(\mathrm{im}(T)) \leq \dim(V)$.

3. Diagonal matrices such that at least one of the entries is 0.

4. Given a basis (v_1, v_2) of V, a linear map $T : V \to V$ can be represented as a 2×2 matrix. Composition of linear maps corresponds to matrix multiplication. There are lots of pairs of 2×2 matrices A, B such that $AB \neq BA$, for example

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

5. The image is a subspace. The union of the two real axes is not a subspace (it contains e_1 and e_2 but does not contain $e_1 + e_2$). Hence there is no such linear map.

6. The preimage of 0 is the kernel. The kernel is a subspace and, as in the last problem, the union of the three coordinate axes is not a subspace.

7. For the second and third matrices, the preimages of vectors in the image form a collection of parallel lines. Since the first matrix is invertible, the preimage of a point is a point.

8. The unit cube is sent to a parallelepiped: a figure with six faces such that opposite faces are parallel. This is the 3-dimensional analog of a parallelogram.

9. The unit cube is mapped to a hexagon in the plane. Some of the sides of the hexagon might have length zero or the hexagon might be squashed onto a line or a point.

10. Given $v \in V$ write $v = u_1 + u_2$. You can do this since $V = U_1 + U_2$. Since $U_1 + U_2$ is a direct sum, u_1 and u_2 are determined by v. Therefore define $T(v)$ to be $T_1(u_1) + T_2(u_2)$. To check linearity: $cv = cu_1 + cu_2$ is the decomposition of cv into its U_1 and U_2 parts so

$$T(cv) = T_1(cu_1) + T_2(cu_2) = cT_1(u_1) + cT_2(u_2) = c(T_1(u_1) + T_2(u_2)) = cT(v)$$

and if $v = u_1 + u_2$ and $w = u_1' + u_2'$ are the decompositions of v and w into their U_1 and U_2 parts then $v + w = (u_1 + u_1') + (u_2 + u_2')$ is the decomposition of $v + w$ into its U_1 and U_2 parts so

$$T(v+w) = T_1(u_1+u_1')+T_2(u_2+u_2') = T_1(u_1)+T_1(u_1')+T_2(u_2)+T_2(u_2')$$

$$= T_1(u_1) + T_2(u_2) + T_1(u_1') + T_2(u_2') = T(v) + T(w).$$

Such a map T is unique: if there were another map S such that $S(u_1) = T_1(u_1)$ and $S(u_2) = T_2(u_2)$ then for $v = u_1+u_2$, then $T(v) = S(u_1+u_2) = S(u_1) + S(u_2) = T_1(u_1) + T_2(u_2) = T(v)$.

11. The zero map, the one that sends all vectors to 0, is the additive inverse in $\mathcal{L}(V, V)$. Hence every subspace of $\mathcal{L}(V, V)$ contains it.

    If $\dim(V) \neq 0$, then the zero map is neither injective nor surjective, so in particular not invertible. Hence the set of invertible elements cannot be a subspace.