Math 110
June 25, 2018
Linear Maps

1. `[Removed]`

2. Let $T : V \to W$. Prove that $\dim \operatorname{im}(T) \leq \dim V$.

3. Let $(v_1, \ldots, v_n)$ be a basis for V. Can you find an linear map $T : V \to V$ which sends each v_i to multiple of itself but which is not invertible? What does the matrix (with respect to $(v_1, \ldots, v_n)$) for such a linear map look like?

4. Let $\dim(V) = 2$. Prove that there exist $T : V \to V$ and $S : V \to V$ such that $T \circ S \neq S \circ T$.

5. Can you find a linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$ whose image is the two coordinate axes of $\mathbb{R}^2$?

6. Let $f : X \to Y$ be a map of sets. Recall that the preimage of $y \in Y$ is the set of all $x \in X$ such that $f(x) = y$. Can you find a linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that the preimage of 0 is the three coordinate axes of $\mathbb{R}^3$?

7. Consider the following matrices as linear maps $\mathbb{R}^2 \to \mathbb{R}^2$:

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

   For each, indicate what the preimages of vectors in $\mathbb{R}^2$ would look like drawn in the plane. Point out the kernel of each map.

8. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$. Suppose T is invertible. What does it do, geometrically, to the unit cube in $\mathbb{R}^3$?

9. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear map. Let C be the unit cube in $\mathbb{R}^3$. Draw some pictures of what $T(C)$ might look like in $\mathbb{R}^2$.

10. Suppose $V = U_1 \oplus U_2$. Let $T_1 : U_1 \to W$ and $T_2 : U_2 \to W$ be linear maps. Show that there exists a unique linear map $T : V \to W$ which restricts to T_1 and T_2 on U_1 and U_2, respectively.

11. Suppose that $\dim(V) \neq 0$. Prove that invertible linear maps $T : V \to V$ do not form a subspace of $\mathcal{L}(V, V)$.