Math 110

June 21, 2018

Bases and Linear Transformations (SOLUTIONS)

1. Suppose for a contradiction that

$$v = a_1 v_{i_1} + \cdots + a_n v_{i_n} = b_1 v_{j_1} + \cdots + b_m v_{j_m}$$

are two ways of expressing $v \in \text{Span}\{v_1, v_2, \ldots\}$ as a linear combination of the elements $\{v_1, v_2, \ldots\}$. By adding some terms with 0 coefficients to each side, one can assume that the two linear combinations use elements from the same finite subset of $\{v_1, v_2, \ldots\}$:

$$v = a_1 v_1 + \cdots + a_n v_n = b_1 v_n + \cdots + b_n v_n.$$

Subtract one side from the other

$$0 = (a_1 - b_1)v_1 + \cdots + (a_n - b_n)v_n.$$

Since the two expressions are different at least one of the coefficients $a_i - b_i$ is nonzero. This contradicts linear independence of the set $\{v_1, \ldots, v_n\}$.

2. The strategy used here is to construct a sequence of linearly independent subsets $B_1 \subset B_2 \subset \cdots \subset B_n$ where B_n will be a basis. Start with $B_1 = \{v_1\}$. Set $B_{i+1} := B_i$ if $B_i \cup v_{i+1}$ is linearly independent. Set $B_{i+1} = B_i$ otherwise.

Note that if $B_i \cup v_{i+1}$ is linearly dependent, then v_{i+1} can be written as a linear combination of $\{v_1, \ldots, v_i\}$. To see this, let

$$a_1 v_1 + \cdots + a_i v_i + a_{i+1} v_{i+1} = 0$$

be a linear relation in $B_i \cup v_{i+1}$. Because B_i is linearly independent, $a_{i+1} \neq 0$. Hence

$$v_{i+1} = -\frac{a_1}{a_{i+1}} v_1 - \cdots - \frac{a_i}{a_{i+1}} v_i. \tag{1}$$

Let B_n be the last set produced in this process. By construction, it is linearly independent. It remains to check that it spans. Because $\{v_1, \ldots, v_n\}$ spans, for any $v \in V$ one can write

$$v = a_1 v_1 + \cdots + a_n v_n.$$

The equation (1) and the argument before it shows that an element v_i that is not in B_n can be written as a linear combination of elements in $B_{i-1} \subset B_n$. Therefore v can be rewritten as a linear combination of vectors in B_n.

Since B_n is linearly independent and spans, it is a basis.[1]

[1]The essential idea in this proof is that if $\{v_1, \ldots, v_m\}$ is linearly independent and adding a vector v to $\{v_1, \ldots, v_m\}$ makes it linearly dependent, then v must necessarily be in $\text{Span}\{v_1, \ldots, v_m\}$.

3. Let $u_1, \ldots, u_m$ be a basis for V. Since the basis spans, so does the set

$$\{v_1, \ldots, v_n, u_1, \ldots, u_m\}.$$

Apply the process from the previous proof to extract a basis for this spanning set. Note that, since you start at v_1 and the first n vectors are linearly independent, $\{v_1, \ldots, v_n\}$ is included in the basis you end up constructing.

In class I suggested the following argument: "Start with $v_1, \ldots, v_n$, add any vector v_{n+1} not in the span of $\{v_1, \ldots, v_n\}$, and repeat this procedure until you have $\dim(V)$ vectors. At each stage you have a linearly independent collection of vectors. A collection of $\dim(V)$ linearly independent vectors is a basis for its span. Any subspace of V has dimension at most $\dim(V)$." The last sentence is true, though the easiest proofs I know all involve citing the result this problem aims to prove.

4. If $b \neq 0$, then

$$f(x+y) = ax + b + ay + b \neq a(x+y) + b = f(x+y).$$

Therefore, in order for f to be linear b needs to be 0. If b is zero, then it is easy to check that f is linear.

5. $T(0) = T(0v) = 0T(v) = 0$.

6. You want to show that $T^{-1}(w_1 + w_2) = T^{-1}(w_1) + T^{-1}(w_2)$. Since T is invertible and hence surjective, $w_1 = T(v_1)$ and $w_2 = T(v_2)$ for some v_1 and v_2. Then

$$T^{-1}(w_1 + w_2) = T^{-1}(T(v_1) + T(v_2)) = T^{-1}(T(v_1 + v_2)) = v_1 + v_2$$
$$= T^{-1}(w_1) + T^{-1}(w_2).$$

Similarly, you want to show that $T^{-1}(cw) = cT^{-1}(w)$. Write $w = T(v)$ then

$$T^{-1}(cw) = T^{-1}(cT(v)) = T^{-1}(T(cv)) = cv = cT^{-1}(w).$$

7. First check that $\{T(v_1), T(v_2), \ldots\}$ spans W. Let $w \in W$ be an arbitrary vector. Write

$$T^{-1}(w) = a_1 v_{i_1} + \cdots + a_n v_{i_n}.$$

Applying T to each side then shows that $w = a_1 T(v_{i_1}) + \cdots + a_n T(v_{i_n})$. Hence $w \in \text{Span}\{T(v_1), T(v_2), \ldots\}$ and so $\{T(v_1), T(v_2), \ldots, \}$ spans all of W.

Similarly, let

$$a_1 T(v_{i_1}) + \cdots + a_n T(v_{i_n}) = 0$$

be a linear relation. Then since $T^{-1}(0) = 0$, applying T gives

$$a_1 v_{i_1} + \cdots + a_n v_{i_n} = 0.$$

Linear independence of the v_is shows that $(a_1, \ldots, a_n) = (0, \ldots, 0)$. Hence $\{T(v_1), T(v_2), \ldots\}$ is linearly independent.

Since $\{T(v_1), T(v_2), \ldots\}$ is linearly independent and spanning, $(T(v_1), T(v_2), \ldots)$ is a basis.

8. This was a bit of a trick question (on purpose) since I wasn't entirely clear about what I meant by integration.

   Differentiation is linear:

$$\frac{d}{dx}(af + bg) = a\frac{df}{dx} + b\frac{dg}{dx}$$

   for a, b constants and f, g differentiable functions.

   Differentiation is not a linear map because it is not injective: all constant functions differentiate to the 0 function.

   Integration defined by "take an antiderivative"

$$f \mapsto \int f$$

   is not even a function, and therefore is not a linear map. Integration over a particular subset $A \subset \mathbb{R}$

$$f \mapsto \int_A f$$

   is a linear map from the (infinite dimensional) vector space of differentiable functions to $\mathbb{R}$.