Math 110

June 19, 2018

Vector Spaces, Linear Independence, Subspaces, Span (SOLUTIONS)

1.

2. Yes they are linearly independent. Suppose for a contradiction that they are not linearly independent. Then there exists $(a_1, a_2) \neq (0, 0)$ such that

$$a_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which contradicts $(a_1, a_2) \neq (0, 0)$.

3. Yes they are linearly independent. Suppose for a contradiction that they are not linearly independent. Then there exists $(a_1, a_2) \neq (0, 0)$ such that

$$a_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a_1 + a_2 \\ a_1 - a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow a_1 + a_2 = 0 \text{ and } a_1 = a_2 \Rightarrow a_1 = a_2 = 0$$

contradicting $(a_1, a_2) \neq (0, 0)$.

4. If $n = m$, then x^n and x^m are linearly dependent. For example, $x^n - x^m = 0$ is linear relation in this case.

   If $n \neq m$, then they are linearly independent. Suppose for a contradiction that they were linearly dependent. Then there exists $(a_1, a_2) \neq (0, 0)$ such that

$$a_1 x^n + a_2 x^m = 0.$$

   This must hold for all $x \in \mathbb{F}$ and implies

$$a_1 x^n = -a_2 x^m.$$

   The left and right sides are different functions if a_1 and a_2 are nonzero and if $n \neq m$. This might seem obvious, but what follows is a rigorous proof. Evalulating each side at $x = 1$ shows that

$$a_1 = -a_2$$

   and evaluating at $x = 2$ shows that

$$a_1 2^n = -a_2 2^m.$$

Since $a_1 = -a_2$ and by assumption the a_is are nonzero, one can divide both sides by $a_1 = -a_2$ to get

$$2^n = 2^m.$$

Since $n \neq m$, this provides a contradiction.

5. Let W be the subset of $\mathbb{F}^n$ consisting of vectors of the form $(a_1, a_2, \ldots, a_{n-1}, 0)$. In order to check that W is a subspace you have to prove three things:

$$0 \in W,\ w_1, w_2 \in W \Rightarrow w_1 + w_2 \in W,\ w \in W \Rightarrow cw \in W.$$

Setting each $a_1 = a_2 = \cdots = a_{n-1} = 0$ shows that $0 \in W$.

$$(a_1, \ldots, a_{n-1}, 0) + (b_1, \ldots, b_{n-1}, 0) = (a_1 + b_1, \ldots, a_{n-1} + b_{n-1}, 0)$$

is of the right form so sums are in W. And similarly

$$(ca_1, \ldots, ca_{n-1}, 0)$$

is of the right form, so scalar multiples are in W.

Let W' be the subset of $\mathbb{F}^n$ consisting of vectors of the form $(a_1, a_2, \ldots, a_{n-1}, 1)$. This is not a subspace since it does not contain zero.

6. Let W_1 and W_2 be subspaces. Let $v, u \in W_1 \cap W_2$. This means that v and u in W_1 and they are in W_2. Since $v, u \in W_1$ then $v + u \in W_1$. Since $v, u \in W_2$ then $v + u \in W_2$. Hence $v + u \in W_1 \cap W_2$.

   Since W_1 and W_2 are subspaces, then $0 \in W_1$ and $0 \in W_2$. Hence $0 \in W_1 \cap W_2$.

   Let $w \in W_1 \cap W_2$. Then $w \in W_1$, which implies $cw \in W_1$. And $w \in W_2$, which implies $cw \in W_2$. Hence $cw \in W_1 \cap W_2$.

   This proves that $W_1 \cap W_2$ satisfies the three requirements of a subspace.

7. Let $w \in W$. Then $(-1)w \in W$. And $(-1)w = -w$ by a proposition proved in class.

8. Suppose that $W_1 \cap W_2 = \{0\}$. Suppose that $w_1 + w_2 = w_1' + w_2'$ are two different ways of writing an element in $W_1 + W_2$ where $w_i, w_i' \in W_i$. Then $w_1 - w_1' = w_2 - w_2'$. Since the two sides are equal, and the right is in W_2, and the left in W_1, both sides are in W_1 and W_2, hence in $W_1 \cap W_2$. Then $w_1 - w_1' = 0$ and $w_2 - w_2' = 0$. Hence $w_1 = w_1'$ and $w_2 = w_2'$.

   Suppose that $w_1 + w_2 = w_1' + w_2' \Rightarrow w_1 = w_1'$ and $w_2 = w_2'$. Let $w \in W_1 \cap W_2$. Then $w + (-w) = 0 + 0 \Rightarrow w = 0$ and $(-w) = 0$. Hence $W_1 \cap W_2$ only contains 0.

9. The first span consists of vectors of the form

$$a_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 \\ 2a_1 \\ 3a_1 \end{pmatrix}, \quad a_1, a_2 \in \mathbb{F}$$

The second consists of the vectors of the form

$$b_1 \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + b_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} b_2 \\ 2b_1 \\ 3b_1 \end{pmatrix}, \quad b_1, b_2 \in \mathbb{F}.$$

Set $b_1 = a_1$ and $b_2 = a_1 + a_2$ to see that the second span is contained in the first. Set $a_1 = b_1$ and $a_2 = b_2 - b_1$ to see that the first span is contained in the second.

10. Since addition wasn't changed you only need to check the axioms involving scalar multiplication. Note that

$$av = 0$$

for all $a \in \mathbb{F}$ and $v \in \mathbb{F}^n$. Hence

$$a(bv) = 0 \text{ and } (ab)v = 0$$

so $a(bv) = (ab)v$. Similarly

$$(a+b)(v+w) = 0 \text{ and } av + bv + aw + bw = 0 + 0 + 0 + 0 = 0$$

so $(a+b)(v+w) = av + bv + aw + bw$.

Suppose for a contradiction that $1v = v$ were implied by the other axioms. Then it would hold under $\mathbb{F}^n$ with this new scalar multiplication, because $\mathbb{F}^n$ with this new scalar multiplication satisfies all the other axioms. But unless $v = 0$, then $1v \neq v$ in $\mathbb{F}^n$ with this new scalar multiplication, giving a contradiction.