1. (a) We can do this using antidifferentiation. $F(x) = \frac{1}{2}x^2$ is an antiderivative for $f(x) = x$, so

$$\int_{-1}^{2} x\,dx = F(x) - F(-1) = \frac{1}{2}(2)^2 - \frac{1}{2}(-1)^2 = \frac{3}{2}$$

(b) The formula for the left Riemann sum with n subintervals of the integral $\int_a^b f(x)\,dx$ is

$$L_n = \Delta x \sum_{i=1}^{n} f\left(a + (i-1)\left(\Delta x\right)\right),$$

where $\Delta x = \frac{b-a}{n}$ is the width of a subinterval. In this case, we have

$$\begin{aligned} L_6 &= \frac{2-(-1)}{6} \sum_{i=1}^{6} f\left(-1 + (i-1)\left(\frac{2-(-1)}{6}\right)\right) \\ &= \frac{1}{2} \sum_{i=1}^{6} f\left(-1 + \frac{i-1}{2}\right) \\ &= \frac{1}{2} \sum_{i=1}^{6} \left(-1 + \frac{i-1}{2}\right) \\ &= \frac{1}{2}\left(-1 + \frac{-1}{2} + 0 + \frac{1}{2} + 1 + \frac{3}{2}\right) \\ &= \frac{3}{4} \end{aligned}$$

(c) The formula for the approximation to $\int_a^b f(x)\,dx$ using the trapezoid rule and n subintervals is

$$T_n = \frac{\Delta x}{2}\left[f(a) + 2f(a+\Delta x) + 2f(a+2\Delta x) + \cdots + 2f(a+(n-1)\Delta x) + f(a+n\Delta x)\right],$$

where $\Delta x = \frac{b-a}{n}$ is the width of a subinterval. In this case, we have

$$\begin{aligned}
T_6 &= \frac{1}{2} \cdot \frac{1}{2} \left[f(-1) + 2f(-0.5) + 2f(0) + 2f(0.5) + 2f(1) + 2f(1.5) + f(2)\right] \\
&= \frac{1}{4}[-1 + 2(-0.5) + 2(0) + 2(0.5) + 2(1) + 2(1.5) + 2] \\
&= \frac{1}{4}[-1 - 1 + 0 + 1 + 2 + 3 + 2] \\
&= \frac{6}{4} = 32
\end{aligned}$$

2. (a) We know that this series converges when $x = 0$, so let's think about $x \neq 0$ and use the ratio test. The ratio test says that to decide whether or not $\sum_{k=0}^{\infty} a_k$ converges, we should compute $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|$. If this limit is < 1, then the series converges. If it is > 1, then the series diverges and if it is $= 0$, then the ratio test is inconclusive. In this case,

$$\begin{aligned}
\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| &= \lim_{k\to\infty} \left|\frac{x^{k+1}}{(k+1)!} \frac{k!}{x^k}\right| \\
&= \lim_{k\to\infty} \left|\frac{x}{k+1}\right| \\
&= 0,
\end{aligned}$$

so the ratio test tells us that the series converges for all $x \neq 0$. This means that the series converges for all x.

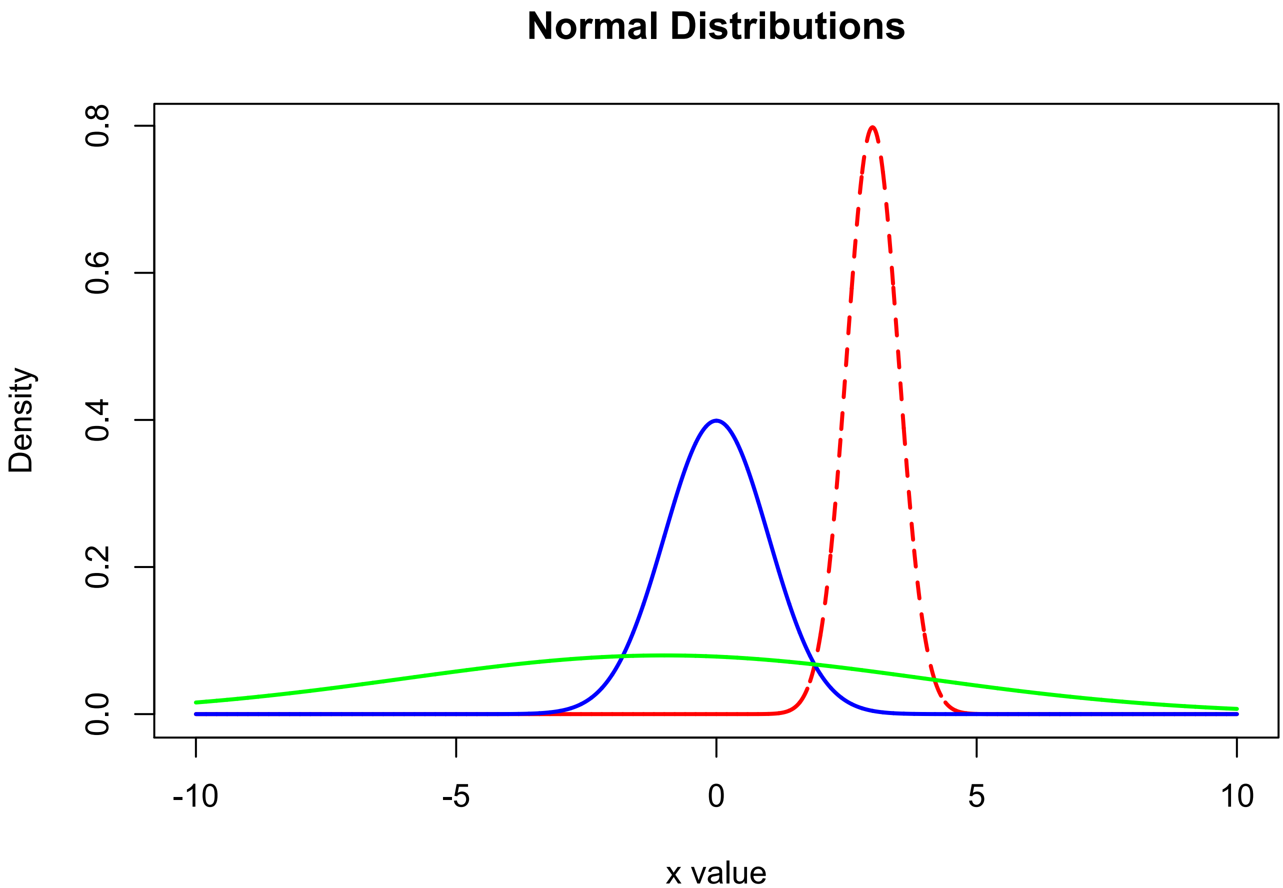
(b) One of the Taylor series that we should probably have committed to memory is that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Plugging in $x = 1$, we see that $e = \sum_{n=0}^{\infty} \frac{1}{n!}$. Since $\sum_{n=0}^{\infty} \frac{1}{n!}$ is (by definition) the limit of the sequence of partial sums, the sequence $\sum_{n=0}^{N} \frac{1}{n!}$ is a sequence of numbers that converges to e as $N \to \infty$. Each $\sum_{n=0}^{N} \frac{1}{n!}$ is a rational number since $\frac{1}{n!}$ is rational for any n and adding rational numbers together gives us another rational number.

3. The graph should look like this. The blue curve is the one with mean 0 and standard deviation 1. The red curve has mean 3 and standard deviation 0.5. The green curve has mean -1 and standard deviation 5. The key things to

notice are that the curves are centered about their means; a higher standard deviation corresponds to a shorter and more spread out graph; and maximum of each curve occurs at the mean.



4. (a) This is one that you should memorize. (You should probably also memorize the derivative of arcsin and arccos.)

$$\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$$

(b) We can make the substitution $u = 1 + x^2$, $du = 2x dx$. This gives us

$$\begin{aligned}\int \frac{x}{1+x^2}\,dx &= \frac{1}{2}\int \frac{2x\,dx}{1+x^2}\\ &= \frac{1}{2}\int \frac{du}{u}\\ &= \frac{1}{2}\ln(u) + C\\ &= \frac{1}{2}\ln(1+x^2) + C\end{aligned}$$

(c) Let's try using integration by parts to get rid of $\ln(x)$. Set $u = \ln(x)$ and $dv = \sqrt{x}dx$, so that $du = \frac{dx}{x}$ and $v = \frac{2}{3}x^{3/2}$. Then,

$$\begin{aligned}\int \sqrt{x}\ln(x)\,dx &= \frac{2}{3}x^{3/2}\ln(x) - \frac{2}{3}\int x^{3/2}\cdot\frac{dx}{x}\\ &= \frac{2}{3}x^{3/2}\ln(x) - \frac{2}{3}\int x^{1/2}\,dx\\ &= \frac{2}{3}x^{3/2}\ln(x) - \frac{4}{9}x^{3/2} + C\end{aligned}$$

(d) Let's use integration by parts to get rid of the x. Set $u = x$ and $dv = \cos(x)\,dx$, so $du = dx$ and $v = \sin(x)$. Then

$$\begin{aligned}\int x\cos(x)\,dx &= x\sin(x) - \int \sin(x)\,dx\\ &= x\sin(x) + \cos(x) + C\end{aligned}$$

5. (a) We have

$$\begin{aligned}\int_1^\infty \frac{1}{x^2}\,dx &= \lim_{a\to\infty}\int_1^a \frac{1}{x^2}\\ &= \lim_{a\to\infty}\left[-\frac{1}{x}\right]_1^a\\ &= \lim_{a\to\infty}(-\frac{1}{a} + \frac{1}{1})\\ &= 1\end{aligned}$$

(b) We have

$$\int_a^b x^c\,dx = \left[\frac{1}{c+1}x^{c+1}\right]_a^b = \frac{1}{c+1}(b^{c+1} - a^{c+1})$$

as long as $c+1 \neq 0$. If $c = -1$, then we have

$$\int_a^b \frac{dx}{x} = [\ln x]_a^b = \ln(b) - \ln(a).$$

6. Remember that we need two things from f for it to be a pdf. We want for $f(x) \geq 0$ for all x and we want for $\int_{-\infty}^{\infty} f(x)\,dx = 1$. To compute the cdf of the pdf that you wrote down, you need to compute $F(x) = \int_{-\infty}^{x} f(t)\,dt$. Here are some examples:

   (a) $f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ and $F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,dx$. In this case, this is the best that we can do for expressing the cdf.

   (b) Let's guess that there's a pdf that has the form

$$f(x) = \begin{cases} Ce^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

   This $f(x) \geq 0$, so we need to see if we can find a constant C such that $\int_{-\infty}^{\infty} f(x)\,dx = 1$. We have

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\,dx &= \int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} Ce^{-\lambda x}\,dx \\ &= -\frac{C}{\lambda}[e^{-\lambda x}]_0^{\infty} \\ &= -\frac{C}{\lambda}(-1) \end{aligned}$$

   so we have to have $C = \lambda$ for the pdf to integrate to 1. Our pdf is then $f(x) = \lambda e^{-\lambda x}$. The cdf is given by $F(x) = 0$ if $x < 0$. If $x > 0$, then

$$\begin{aligned} F(x) &= \int_0^x \lambda e^{-\lambda t}\,dt \\ &= [-e^{-\lambda t}]_0^x \\ &= -e^{-\lambda x} + 1 \end{aligned}$$

(c) For a last example, let's take

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

In this case the cdf is given by

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$$