Math 10a

Practice Midterm 2 #2

1. In summation notation, write down the left Riemann sum estimate for $\int_0^1 x(1-x)dx$ using 1000 intervals.

$$\frac{1}{1000}\sum_{k=0}^{999}\frac{k}{1000}\left(1-\frac{k}{1000}\right).$$

2. (a) What is the Taylor series for $\ln(x)$ centered at $x=1$?

Differentiating $f(x)=\ln(x)$:

$$f'(x)=\frac{1}{x}$$
$$f''(x)=-\frac{1}{x^2}$$
$$f'''(x)=\frac{2}{x^3}$$
$$f^{(4)}(x)=-\frac{3\cdot 2}{x^4}$$
$$f^{(5)}(x)=-\frac{4\cdot 3\cdot 2}{x^5}.$$

We note a pattern

$$f^{(k)}(x)=(-1)^{k+1}\frac{(k-1)!}{x^k},\ k\geq 1.$$
$$f^{(k)}(1)=(-1)^{k+1}(k-1)!,\ k\geq 1.$$

Hence

$$f(x)=\sum_{k=0}^{\infty}f^{(k)}(1)\frac{(x-1)^k}{k!}=\ln(1)+\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(k-1)!(x-1)^k}{k!}=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(x-1)^k}{k}.$$

Note that we have to be careful to split off the $k=0$ term, since our formula for the derivative doesn't work for $k=0$.

(b) What is the radius of the convergence of the series from part (a)?

I see factorials, so I perform the ratio test:

$$\left|\frac{a_{k+1}}{a_k}\right|=\left|\frac{\frac{(-1)^{k+2}(x-1)^{k+1}}{k+1}}{\frac{(-1)^{k+1}(x-1)^k}{k}}\right|=\left|\frac{k(x-1)}{k+1}\right|\to|x-1|$$

so the series converges for $|x-1|<1$, diverges for $|x-1|>1$, so the radius of convergence is 1.

(c) Write down a series of rational numbers converging to ln(1/3).

Since the series coverges for $x = \frac{1}{3}$,

$$\ln\left(\frac{1}{3}\right) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\left(\frac{1}{3}-1\right)^k}{k} = \sum_{k=1}^{\infty} -\frac{2^k}{3^k k}.$$

3. (a) Suppose giraffe neck lengths are normally distributed with mean of 6 feet and a standard deviation of 6 inches. What is the probability, given a randomly selected giraffe, that its neck is shorter than 5 feet?

   5 feet is two standard deviations below the mean, and the area under the bell curve to the left of -2 standard deviations is .025. Hence the answer is 2.5%.

   (b) Suppose giraffe tongue lengths are normally distributed with a mean of 20 inches (!) and a standard deviation of 3 inches. What is the probability that a randomly selected giraffe will have a tongue of length between 20 and 23 inches?

   20 is the mean and 23 is one standard deviation above the mean. Hence we're looking at the area under a bell curve between the mean and one standard deviation above the mean. This is half the area between ± 1 standard deviations, hence is 34%.

4. Compute the following integrals:

   (a)

$$\int \frac{x}{1-x} dx$$

   $u = 1 - x$, $du = -dx$, $x = 1 - u$

$$\int \frac{1-u}{u}(-du) = \int \left(1 - \frac{1}{u}\right) du = u - \ln|u| + C = 1 - x - \ln|1-x| + C$$

   (b)

$$\int x\sqrt{x+1}dx$$

   $u = x + 1,\ du = dx, x = u - 1$

$$\int (u-1)\sqrt{u}du = \int u^{3/2} - u^{1/2} = \frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2} = \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C.$$

   (c)

$$\int e^x \sin(x)dx$$

   See the lecture slides.

(d)

$$\int \sin(\sqrt{x})dx$$

$u = \sqrt{x},\ du = \frac{1}{2}\frac{1}{\sqrt{x}}dx = \frac{dx}{2u} \Rightarrow dx = 2udu$

$$= 2\int u\sin(u)du = 2\int u\frac{d}{du}(-\cos(u))du$$

$$= -2u\cos(u)+2\int\cos(u) = -2u\cos(u)+2\sin(u)+C = -2\sqrt{x}\cos(\sqrt{x})+2\sin(\sqrt{x})+C.$$

5. Compute the following integrals:

(a)

$$\int_1^2 \frac{x}{\sqrt{1+x^2}}dx$$

$u = 1+x^2,\ du = 2xdx\frac{1}{2}\int_{x=1}^{x=2}\frac{du}{\sqrt{u}} = \frac{1}{2}\int_{u=2}^{u=5}\frac{du}{\sqrt{u}} = \frac{1}{2}\frac{u^{1/2}}{1/2}\Big|_{u=2}^{u=5} = \sqrt{5}-\sqrt{2}.$

(b)

$$\int_0^\pi x\sin(x)dx$$

$$\int_0^\pi x\left(\frac{d}{dx}-\cos(x)\right)dx = -x\cos(x)\Big|_0^\pi + 2\int_0^\pi \cos(x)dx = \pi.$$

6. Recall that the uniform distribution from 0 to 1 is defined to be one whose pdf is

$$f(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}.$$

What is the cdf of this uniform distribution? Sketch a graph.

I'll leave it to you to sketch the graph. The cdf $F(x)$ is the area under the pdf to the left of x, so

$$F(x) = \int_{-\infty}^x f(t)dt = \int_{-\infty}^x \begin{cases} 0 & t<0 \\ 1 & 0\le t\le 1\ dt \\ 0 & t>1 \end{cases}$$

$$= \begin{cases} \int_{-\infty}^x 0dt & x<0 \\ \int_{-\infty}^0 0dt + \int_0^x 1dt & 0\le x\le 1 \\ \int_{-\infty}^0 0dt + \int_0^1 1dt + \int_1^x 0dt & x>1 \end{cases} = \begin{cases} 0 & x<0 \\ x & 0\le x\le 1 \\ 1 & x>1 \end{cases}.$$